Multiple Antennas in Wireless Communications

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Outline

- MIMO
- Capacity of MIMO systems
  - channel known at the transmitter and receiver
  - channel known at the receiver
A MIMO channel is represented by a channel matrix
MIMO
An overview

MIMO channels arise in many different scenarios

- wireline systems
- wireless systems

Some examples

- Frequency-Selective Channels
- Multicarrier Channels
- Multi-Antenna Channels
- ...
MIMO
Frequency-Selective Channels

Convolutional channel

\[ x(k) \xrightarrow{h(k)} y(k) \]

MIMO representation

\[
\begin{bmatrix}
  y(0) \\
  y(N-1) \\
  y(N) \\
  y(N+L-1)
\end{bmatrix}
= \begin{bmatrix}
  h(L) & \cdots & h(0) & 0 & \cdots & 0 \\
  0 & \ddots & \ddots & \ddots & 0 & \ddots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & \cdots & 0 & h(L) & h(0) & \ddots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & \cdots & 0 & h(L) & h(0) & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
  x(-L) \\
  \vdots \\
  x(-1) \\
  x(0) \\
  \vdots \\
  x(N-1) \\
  x(N) \\
  \vdots \\
  x(N+L-1)
\end{bmatrix}
\]
MIMO
Multicarrier Channels - OFDM

\[ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \rightarrow \text{Add CP} \rightarrow h(n) \rightarrow \text{Discard CP} \rightarrow y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \]

Thermal noise
The received samples $y$ take the form

$$y = Hx + n$$

with

$$H = \begin{bmatrix}
h(0) & 0 & \cdots & \cdots & 0 & h(L) & \cdots & h(1) \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \ddots & h(0) & \ddots & \ddots & \ddots & \ddots & \vdots \\
h(L) & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & h(0) & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & h(L) & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\
0 & \cdots & \cdots & 0 & h(L) & \cdots & \cdots & h(0) \\
\end{bmatrix}$$
MIMO
Multi-Antenna Channels

\[ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{bmatrix} \xrightarrow{H} \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix} \quad \Rightarrow \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \]
MIMO
Multi-Antenna Channels

- Basic narrow-band system model

\[ y = Hx + n \]

with

\[
H = \begin{bmatrix}
H_{1,1} & H_{1,2} & \cdots & H_{1,M} \\
H_{2,1} & H_{2,2} & & H_{2,1} \\
\vdots & \vdots & \ddots & \vdots \\
H_{N,1} & H_{N,2} & \cdots & H_{N,M}
\end{bmatrix}
\]

- In broad-band channel matrix entries are frequency (delay) dependent
MIMO
Multi-Antenna Channels

Advantages

- no additional bandwidth or time
- array gain
- diversity gain

Array gain
  - MISO-SIMO-MIMO

Diversity gain
  - MISO-SIMO-MIMO
Capacity

Shannon capacity - Definition

Claude Shannon (April 30, 1916 – February 24, 2001)

Channel capacity was pioneered by Claude Shannon back in 1948 [Shannon, 1948]

- using a mathematical theory of communication
- the central and most famous success of information theory
Capacity
Shannon capacity - Definition

The capacity of a channel is operationally defined as

- the maximum data rate at which reliable communication can be performed
- without any constraint on transmitter and receiver complexity

Denote by $C$ the channel capacity. Then, for any rate $R < C$

- there exist rate $R$ channel codes with arbitrarily low error probability
Capacity

Shannon capacity - Definition

Although theoretically possible to communicate at $R < C$

- code design is a very difficult problem!
- reasonable block length and encoding/decoding complexity

Tremendous progress have been made for single-antenna Gaussian channels

- TURBO codes, LDPC codes, ... 

Such solutions do not apply to multi-antenna Gaussian channels

- due to the spatial dimension
Capacity of single antenna Gaussian channels

Shannon capacity

The capacity is mathematically defined as

\[
C = \max_{f(x), \mathbb{E}\{\|x\|^2 \leq P\}} \mathcal{I}(x, y)
\]

where \(f(x)\) denotes the input distribution while

\[
\mathcal{I}(x, y) = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log \frac{p(x, y)}{p(x)p(y)}
\]

is the mutual information
The capacity of a single antenna Gaussian channel is given by

\[ C = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} (y - x)^2} dy \]
The capacity of a single antenna Gaussian channel is given by

\[ C = \log(1 + SNR) \text{ bit/s/Hz} \]
Capacity of single antenna Gaussian channels

Shannon capacity

- Observe that when $SNR \approx 0$

$$C \approx SNR \log e$$

the capacity increases linearly with received power

- When $SNR \gg 1$

$$C \approx \log SNR$$

the capacity increases logarithmically with received power
Capacity of single antenna Gaussian channels

Shannon capacity
Capacity of single antenna Gaussian channels

Shannon capacity
Capacity of MIMO channels

MIMO flat fading model

\[
y = Hx + n
\]  

(1)
Capacity of MIMO channels

Operating conditions

1. **H** is *deterministic*
   - known at the transmitter and receiver

2. **H** is *random* with a given probability distribution
   - known at the transmitter and receiver (slow-fading)
   - known at the receiver (fast-fading)
Capacity of MIMO channels

Deterministic channel - Via singular value decomposition

Computing the singular vector decomposition of $H$ produces

$$H = U \Lambda V^H$$

- $U \in \mathbb{C}^{N \times N}$ and $V \in \mathbb{C}^{M \times M}$ are unitary matrices
- $\Lambda \in \mathbb{C}^{N \times M}$ is diagonal \(^1\) with $K$ non-negative elements
- $K = \text{rank}(H) \leq \min(M, N)$

Define

$$\tilde{x} = V^H x$$

$$\tilde{y} = U^H y$$

$$\tilde{n} = U^H n$$

\(^1\)We call this matrix diagonal even though it may not be square
Capacity of MIMO channels

Deterministic channel - Via singular value decomposition

Pre-processing

Channel

Post-processing
Capacity of MIMO channels

Deterministic channel - Via singular value decomposition

Then, we may write (1) as

\[ \tilde{y} = \Lambda \tilde{x} + \tilde{n} \]  \hspace{1cm} (2)

or, equivalently,

\[ \tilde{y}_k = \sqrt{\lambda_k} \tilde{x}_k + \tilde{n}_k \hspace{1cm} k = 1, 2, \ldots, K \]  \hspace{1cm} (3)

The MIMO channel has been decomposed into \( K \) independent parallel Gaussian channels.
Capacity of MIMO channels

Deterministic channel - Via singular value decomposition

\[ \tilde{x}_1 \rightarrow \sqrt{\lambda_1} \rightarrow \tilde{y}_1 \]

\[ \tilde{x}_2 \rightarrow \sqrt{\lambda_2} \rightarrow \tilde{y}_2 \]

\[ \vdots \]

\[ \tilde{x}_K \rightarrow \sqrt{\lambda_K} \rightarrow \tilde{y}_K \]
Capacity of MIMO channels

Deterministic channel - Via singular value decomposition

The capacity is thus given by

\[ C = \sum_{k=1}^{K} \log (1 + P_k \lambda_k) \text{ bit/s/Hz} \]  \hspace{1cm} (4)

where \( P_k \) is the power allocated to \( x_k \) such that

\[ \sum_{k=1}^{K} P_k \leq P \]  \hspace{1cm} (5)
Capacity of MIMO channels

Deterministic channel - Unknown at the transmitter

If no channel state information is available at the transmitter

\[ P_k = \frac{P}{K} \tag{6} \]

and the capacity takes the form

\[ C = \sum_{k=1}^{K} \log \left( 1 + \frac{P\lambda_k}{K} \right) \text{ bit/s/Hz} \tag{7} \]
Capacity of MIMO channels
Analog with OFDM

\[ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \rightarrow \text{Add CP} \rightarrow h(n) \rightarrow \text{Discard CP} \rightarrow y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \]

Thermal noise
Capacity of MIMO channels

Analogy with OFDM

The received samples $\mathbf{y}$ take the form

$$\mathbf{y} = \mathbf{Hx} + \mathbf{n}$$

with

$$\mathbf{H} = \begin{bmatrix}
h(0) & 0 & \cdots & \cdots & 0 & h(L) & \cdots & h(1) \\
\vdots & \ddots & \ddots & \cdots & \vdots & 0 & \ddots & \\
\vdots & \ddots & h(0) & \ddots & \ddots & \ddots & 0 & \\
h(L) & \ddots & \ddots & 0 & 0 & 0 & \cdots & \\
0 & \ddots & \ddots & \ddots & h(0) & \ddots & \cdots & \\
\vdots & \ddots & h(L) & \ddots & \ddots & \ddots & \cdots & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \\
0 & \cdots & \cdots & 0 & h(L) & \cdots & \cdots & h(0)
\end{bmatrix}$$
Capacity of MIMO channels

Analogy with OFDM

Observing that

- $H$ in (8) is circulant
- The eigenvalue decomposition of circulant matrices is

$$H = F^H D F$$

where $F$ is the unitary discrete Fourier transform matrix and

$$D = \text{diag} \{ H(k); k = 1, 2, \ldots, N - 1 \}$$

while

$$H(k) = \sum_{\ell=0}^{L-1} h(\ell) e^{-j2\pi\ell(k-1)/N}$$

is the channel frequency response over the $n$th subcarrier.
Capacity of MIMO channels

Analogy with OFDM

Define

\[ x = F^H \tilde{x} \]
\[ y = F\tilde{y} \]
\[ n = F\tilde{n} \]

Then

\[ \tilde{y}(k) = H(k)\tilde{x}(k) + \tilde{n}(k) \quad k = 1, 2, \ldots, N \]

The original frequency selective channel has been decomposed into a set of \( N \) independent parallel channels.
Capacity of MIMO channels
Analogy with OFDM

\[
\tilde{x} = \begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\vdots \\
\tilde{x}_N
\end{bmatrix}
\rightarrow \text{IFFT} \rightarrow \text{Add CP} \rightarrow h(n) \rightarrow \text{Discard CP} \rightarrow \text{FFT} \rightarrow \tilde{y} = \begin{bmatrix}
\tilde{y}_1 \\
\tilde{y}_2 \\
\vdots \\
\tilde{y}_N
\end{bmatrix}
\]

Thermal noise

Channel frequency response
Exercise. Take $K = M = N$ and assume $H = I_K$. Determine the channel capacity. Verify that the capacity tends to the limiting value $\log(e)$ as $K \to \infty$. 
Exercise. Assume that the channel power is fixed and equal to \( \sum_{k=1}^{K} \lambda_k = \rho \). Verify that (7) is maximized when \( \mathbf{H} \) is orthogonal.
Capacity of MIMO channels

Deterministic channel - Known at the transmitter

If channel state information is available at the transmitter, we obtain

\[
\mathcal{C} = \max_{\{\tilde{P}_k \geq 0, \sum_{k=1}^{K} \tilde{P}_k \leq P\}} \sum_{k=1}^{K} \log \left(1 + \tilde{P}_k \lambda_k \right) \quad \text{bit/s/Hz} \quad (8)
\]

The solution is found to be

\[
P_k = \left(\mu - \frac{1}{\lambda_k} \right)_+ \quad (9)
\]

where \(\mu\) is chosen to satisfy (5) and \((x)_+ = \max(x, 0)\).
Capacity of MIMO channels
Deterministic channel - Known at the transmitter

Water-filling or water-pouring algorithm

- **Initialization:**
  1. Set $k = 1$

- **Power allocation**
  repeat the following procedure until $P_k \geq 0 \ \forall k$:

  1. $\mu = \frac{P}{K-k+1} \left( 1 + \sum_{i=1}^{K-k+1} \frac{1}{\lambda_i} \right)$
  2. $P_i = \mu - \frac{1}{\lambda_i} \quad i = 1, 2, \ldots, K - k + 1$
  3. if $P_{K-k+1} < 0$ then let $P_{K-k+1} = 0$
  4. $k = k + 1$
Capacity of MIMO channels
Deterministic channel - Known at the transmitter

\[ \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3} \]

\( P_1, P_2, P_3 \)

used eigenmodes

unused eigenmodes
Capacity of MIMO channels
Deterministic channel - Known at the transmitter

**Question.** What is the *near* optimal power allocation strategy for *large* values of $P$?
Capacity of MIMO channels
Deterministic channel - Known at the transmitter

**Question.** What is the *near* optimal power allocation strategy for *large* values of $P$?

![Diagram showing optimal and near-optimal power allocation]

- $P_k = \frac{P}{K}$ for $k = 1, 2, \ldots, K$. 
Capacity of MIMO channels

Deterministic channel - Known at the transmitter

**Question.** What is the *near* optimal power allocation strategy for *small* values of $P$?
Capacity of MIMO channels

Deterministic channel - Known at the transmitter

**Question.** What is the *near* optimal power allocation strategy for *small* values of $P$?

![Graph showing optimal and near optimal power allocation strategies]

- $P_k = P$ for $k = \max_i \lambda_i$. 

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MIMO  
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Capacity of MIMO channels

Deterministic channel - Known at the transmitter

- For large values of $P$

$$
C = \sum_{k=1}^{K} \log \left( 1 + \frac{P}{K} \lambda_k \right) \approx K \log P + \mathcal{O}(1) \quad (10)
$$

with

$$
\mathcal{O}(1) = \sum_{k=1}^{K} \log \left( \frac{\lambda_k}{K} \right)
$$

- For small values of $P$

$$
C \approx P \log e \left( \max_i \lambda_i \right) \quad (11)
$$
Capacity of MIMO channels

Deterministic channel - Known at the transmitter

- Assume $P \gg 1$ and use Jensens’ inequality

\[
\frac{1}{K} \sum_{k=1}^{K} \log \left( \frac{P}{K} \lambda_k \right) \leq \log \left( \frac{P}{K} \left( \sum_{k=1}^{K} \frac{1}{K} \lambda_k \right) \right)
\]

- Define the channel power as

\[
\sum_{k=1}^{K} \lambda_k = \text{tr}\{H^H H\} = \sum_{n=1}^{N} \sum_{m=1}^{M} |H_{n,m}|^2
\]
Capacity of MIMO channels

Deterministic channel - Known at the transmitter

- Assume $P \gg 1$ and use Jensens’ inequality

\[
\frac{1}{K} \sum_{k=1}^{K} \log \left( \frac{P}{K} \lambda_k \right) \leq \log \left( \frac{P}{K} \left( \sum_{k=1}^{K} \frac{1}{K} \lambda_k \right) \right)
\]

- Define the channel power as

\[
\sum_{k=1}^{K} \lambda_k = \text{tr}\{HH^H\} = \sum_{n=1}^{N} \sum_{m=1}^{M} |H_{n,m}|^2
\]

- Well-conditioned channel matrices facilitate communication at high SNR
Capacity of SIMO channels

Deterministic channel

Assume $M = 1$ (single transmit antenna), i.e., $\mathbf{H} = \mathbf{h} \in \mathbb{C}^{N \times 1}$

- single eigenmode $\lambda_1 = \mathbf{h}^H \mathbf{h} = \|\mathbf{h}\|^2$

Then

1. Channel unknown at the transmitter

   $$C = \log \left( 1 + P \|\mathbf{h}\|^2 \right)$$

2. Channel known at the transmitter

   $$C = \log \left( 1 + P \|\mathbf{h}\|^2 \right)$$

Assuming that $\|\mathbf{h}\|^2 = N$ we get

$$C = \log (1 + PN)$$
Capacity of MISO channels

Deterministic channel

Assume $N = 1$ (single receive antenna), i.e., $\mathbf{H} = \mathbf{h} \in \mathbb{C}^{1 \times M}$

- single eigenmode $\lambda_1 = \mathbf{h}^H \mathbf{h} = \|\mathbf{h}\|^2$

Then

1. Channel unknown at the transmitter

$$C = \log \left( 1 + \frac{P}{M} \|\mathbf{h}\|^2 \right)$$

2. Channel known at the transmitter

$$C = \log \left( 1 + P \|\mathbf{h}\|^2 \right)$$

Assuming that $\|\mathbf{h}\|^2 = M$ we get

$$C = \log (1 + PM)$$
Capacity of MIMO channels

Deterministic channel - Known at the transmitter

Summarizing (assume full rank channel matrix)

1. SIMO

\[ C \approx \log (PN) \]

2. MISO

\[ C \approx \log (PM) \]

3. MIMO

\[ C \approx K \log (P) = \min(M, N) \log (P) \]
Capacity of MIMO channels
Spatial-Multiplexing gain

Using multiple transmit and receive antennas
- rate is increased by a factor $\min(M, N)$
- no additional power consumption is required

Such a gain is known as *multiplexing gain*
- or better *spatial multiplexing gain*
Capacity of MIMO channels
Spatial-Multiplexing gain

Spatial - Multiplexing gain
  • MIMO

Array gain
  • MISO-SIMO-MIMO

Diversity gain
  • MISO-SIMO-MIMO
MIMO

Tradeoff between gains on MIMO channels

Array and diversity gains

- basic concept: coherent combination of multiple signals
- multiple receive antennas: they may be simultaneously achieved
- multiple transmit antennas:
  - array gain requires channel knowledge
  - diversity gain does not

Array and spatial-multiplexing gains

- maximum array gain: maximum singular value should be used [Andersen, 2000]
- maximum spatial-multiplexing gain: water-filling over singular values [Telatar, 1999]
MIMO
Tradeoff between gains on MIMO channels

Diversity and spatial-multiplexing gains
- traditionally the design has been focused on either
  - maximum diversity gain [Tarokh, 1998]
  - maximum spatial-multiplexing gain [Telatar, 1999]

There exists a fundamental trade-off [Zheng, 2003]
- spatial-multiplexing gain is related to data rate
- diversity gain is related to error rate
- diversity-multiplexing tradeoff is essentially tradeoff between error rate and data rate
Capacity of single antenna Gaussian channels

Shannon capacity

The capacity is mathematically defined as

\[ C = \max_{f(x), \mathbb{E}\{\|x\|^2\} \leq P} \mathcal{I}(x, y) \]  

(12)

where \( f(x) \) denotes the input distribution while

\[ I(x, y) = \sum_{x \in X} \sum_{y \in Y} p(x, y) \log \frac{p(x, y)}{p(x)p(y)} \]  

(13)

is the mutual information
Capacity of single antenna Gaussian channels

Shannon capacity

Recall that the entropy is given

$$\mathcal{H}(y) = - \sum_{y \in \mathcal{Y}} p(y) \log p(y)$$

while the conditional entropy takes the form

$$\mathcal{H}(y|x) = - \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log p(y|x)$$
The mutual information can be written as

\[ I(x, y) = H(y) - H(y|x) \]

Since \( x \) and \( n \) are independent \( H(y|x) = H(n) \)

\[ I(x, y) = H(y) - H(n) \]

Maximizing \( I(x, y) \) reduces to maximizing \( H(y) \)
Capacity of MIMO channels
Deterministic channel - Alternative derivation

For this purpose, we recall that
- $x$ is zero-mean with covariance $E\{xx^H\} = Q$
- $x$ is independent from $n$

Then
- $y$ is zero-mean with covariance

$$E\{yy^H\} = I_N + HQH^H$$
Capacity of MIMO channels
Deterministic channel - Alternative derivation

**Lemma (1)**

If $\mathbf{z} \in \mathbb{C}^n$ with zero mean and satisfying $\mathbb{E}\{\mathbf{z}\mathbf{z}^H\} = \mathbf{A}$. Then the entropy of $\mathbf{z}$ satisfies $H(\mathbf{z}) \leq \log \det(\pi e \mathbf{A})$ with equality if and only if $\mathbf{z}$ is circularly symmetric complex Gaussian with

$$\mathbb{E}\{\mathbf{z}\mathbf{z}^H\} = \mathbf{A}$$

From the above Lemma, it follows that

- $H(\mathbf{y})$ is maximized when $\mathbf{y}$ is circularly symmetric complex Gaussian

When does it happen?
Capacity of MIMO channels
Deterministic channel - Alternative derivation

Lemma (2)

If $x \in \mathbb{C}^M$ is circularly symmetric complex Gaussian then $Hx$ is circularly symmetric complex Gaussian for any $H \in \mathbb{C}^{N \times M}$.

Lemma (3)

If $x$ and $n$ are independent circularly symmetric complex Gaussians, then $y = Hx + n$ is circularly symmetric complex Gaussian.

Then, the answer to the question ”When does it happen?” is:
- when $x$ is circularly symmetric complex Gaussian
Capacity of MIMO channels
Deterministic channel - Alternative derivation

Collecting all the above facts together, we get

\[ I(x, y) = H(y) - H(n) \approx \log \det (I_N + H^H QH) \]

Using \( \det (I_N + H^H QH) = \det (I_M + HH^H Q) \) yields

\[ I(x, y) \approx \log \det (I_M + HH^H Q) \]

Computing the SVD, i.e., \( HH^H = U\Lambda\Lambda^H U^H \), produces

\[ I(x, y) \approx \det (I_N + \Lambda^H Q\Lambda^H) \]
Capacity of MIMO channels
Deterministic channel - Alternative derivation

We are now left with

\[ C = \max_{Q, \text{tr}\{Q\} \leq P} \det (I_N + \Lambda^H Q \Lambda^H) \]

Using \( \det (A) \leq \prod_{k=1}^{K} [A]_{k,k} \) we eventually obtain

\[ C = \max_{\{\tilde{P}_k \geq 0, \sum_{k=1}^{K} \tilde{P}_k \leq P\}} \log \left( 1 + \tilde{P}_k \lambda_k \right) \]  \hspace{1cm} (14)

with \( \tilde{P}_k = [Q]_{k,k} \)

*The result in (14) is equivalent to (4) obtained via SVD!*
Capacity of MIMO channels

Random channel

Assume that $\mathbf{H}$ is a *random* matrix

- varying independently at each channel use
- independent of $\mathbf{x}$ and $\mathbf{n}$

Assume $\mathbf{H}$

1. known at the transmitter and receiver (slow-fading)
2. known at the receiver (fast-fading)
Capacity of MIMO channels
Random channel - Preliminaries

10% Outage capacity

Ergodic capacity
Capacity of MIMO channels

Random channel - Known at the transmitter

When channel information is available at the transmitter

$$C = E_{\tilde{H}} \left\{ \max_{Q \geq 0, \text{tr}(Q) \leq P} \log \det \left( I + \tilde{H}Q\tilde{H}^H \right) \right\}$$

The solution is found to be water-filling for each $\tilde{H}$

Note that the channel capacity in (??) is valid for any fading distribution
Capacity of MIMO channels
Random channel - Unknown at the transmitter

When no channel information is available at the transmitter

\[ C = \max_{f(x)} \mathcal{I}(x; y, H) \]  \hspace{1cm} (15)

with

\[ \mathcal{I}(x; y, H) = \mathcal{I}(x; H) + \mathcal{I}(x; y|H) = \mathcal{I}(x; y|H) \]  \hspace{1cm} (16)

and

\[ \mathcal{I}(x; y|H) = E_{\tilde{H}} \{ \mathcal{I}(x; y|H = \tilde{H}) \} \]  \hspace{1cm} (17)
Capacity of MIMO channels

Random channel

Substituting (16) and (17) into (15) produces

\[
C = \max_{f(x)} E_{\tilde{H}} \left\{ \mathcal{I}(x; y|H = \tilde{H}) \right\}
\]  

(18)

Theorem (1)

The capacity of the channel is achieved when \( x \) is circularly symmetric complex Gaussian with zero-mean and covariance \( Q = \frac{P}{M} I_M \). The capacity is then given by

\[
C = E_{\tilde{H}} \left\{ \log \det \left( I_N + \frac{P}{M} \tilde{H}\tilde{H}^H \right) \right\}
\]  

(19)
Capacity of MIMO channels

Rayleigh fading model

- The direct computation of (19) is reported in [Telatar, 1999] while further results are illustrated in [Shin, 2003]
Capacity of MIMO channels
Rayleigh fading model

- The direct computation of (19) is reported in [Telatar, 1999] while further results are illustrated in [Shin, 2003]

- It is found that

\[
C = \log(e) \frac{m!}{(n-1)!} \sum_{\ell=0}^{m-1} \sum_{\mu=0}^{m} \sum_{p=0}^{\ell+\mu+n-m} \frac{(-1)^{\ell+\mu}(\ell+\mu+n-m)!}{\ell!\mu!} e^{M/P} F_{p+1}(M/P) \\
\cdot \left[ \binom{n-1}{m-1-\ell} \binom{n}{m-1-\mu} - \binom{n-1}{m-2-\ell} \binom{n}{m-\mu} \right]
\]

where \( m = \min(M, N) \) and \( n = \max(M, N) \) while

\[
F_i(x) = \int_{1}^{\infty} e^{-xy} y^{-i} dy
\]
Capacity of MIMO channels

Simulation results

![Graph showing the relationship between SNR and ergodic capacity for channel known and channel unknown.](image-url)
Capacity of MIMO channels

Rayleigh fading model

Consider some special cases for gaining intuition

- increase SNR
- increase transmit antennas
- increase receive antennas
Capacity of MIMO channels

Scaling laws

- If $N$ and $M$ are fixed and $P$ increases

\[
C \approx \min(M, N) \log(P) + O(1) \tag{20}
\]

The ergodic capacity has a multiplexing gain of $\min(M, N)$

- Each 3 dB of SNR leads to an increase of $\min(M, N)$ bit/s/Hz in spectral efficiency

- This is achieved without channel knowledge at the transmitter
Capacity of MIMO channels
Scaling laws

![Graph showing the relationship between ergodic capacity and SNR for different MIMO configurations](image-url)
Capacity of MIMO channels

Scaling laws

- If $N$ and $P$ are fixed and $M \to \infty$

\[
C = N \log (1 + P)
\]

The capacity is bounded in $M$ and converges to the above result

- This is due to the fact the same amount of power $P$ is divided between more and more antennas
Capacity of MIMO channels

Scaling laws

- If $M$ and $P$ are fixed and $N \to \infty$

$$C = M \log \left(1 + \frac{P}{MN}\right)$$ (21)

The capacity increases approximately as $\log(N)$

- Adding more receive antennas increases the amount of power
  - adding transmit antennas does not
Capacity of MIMO channels

Scaling laws

- If $M = N \to \infty$ and $P$ is fixed

$$C \approx \mathcal{K} \min (M, N)$$  \hspace{1cm} (22)

where $\mathcal{K}$ is a constant depending on the ratio of $M$ and $N$

- The capacity grows linearly with increases $\min (M, N)$
Capacity of MIMO channels

Scaling laws

![Graph showing the relationship between ergodic capacity and \( k \) for different values of \( M \) and \( N \). The graph illustrates how the capacity increases with increasing values of \( k \). The scaling laws are indicated as follows:

- \( M = N = k \)
- \( M = 1, N = k \)
- \( M = k, N = 1 \)
Capacity of MIMO channels

Random channel - Spatially correlated

Assume that $\mathbf{H}$ is random
- with correlated entries

\[
\mathbf{H} = \mathbf{T}^{1/2} \mathbf{W} \mathbf{R}^{1/2}
\]

In such a case the capacity is known only for some special cases.

A lower bound can be computed for $\mathbf{x} \in \mathcal{N}(0, P/M \mathbf{I}_M)$

\[
\mathcal{C} \geq \mathbb{E}_{\tilde{\mathbf{W}}} \left\{ \log \det \left( \mathbf{I}_N + \frac{P}{M} \tilde{\mathbf{W}} \tilde{\mathbf{T}} \tilde{\mathbf{W}} \mathbf{R} \right) \right\} \quad (23)
\]
Capacity of MIMO channels

Random channel - Spatially correlated

Assume $M = N$ and $P \to \infty$

$$C \approx K \log \frac{P}{M} + \log \det (TR) \quad (24)$$

Since

$$- \frac{\log \det (TR)}{M} \geq 0$$

we have that

- spatial correlation degrades system performance
- the linear growth with respect to $K$ is preserved

The above result can be extended to the case $M \neq N$. 
Capacity of MIMO channels

Random channel - Spatially correlated

Loss due to the correlation

Ergodic capacity vs. SNR, dB
Capacity of MIMO channels
Random channel - Rice model

Assume that \( H \) is *random*

- with a line-of-sight component

\[
H = \sqrt{\frac{r}{1 + r}} \bar{H} + \sqrt{\frac{1}{1 + r}} W
\]

Assume for example

\[
\bar{H} = H_A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\]

\[
\bar{H} = H_B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}
\]
Capacity of MIMO channels
Random channel - Rice model

\[ \text{Ergodic capacity} \]

\[ r \]

\[ H_A \quad H_B \]
References


