On the design of Amplify-and-Forward MIMO-OFDM Relay Systems with QoS requirements

Luca Sanguinetti, Member, IEEE, Antonio A. D’Amico and Yue Rong, Senior Member, IEEE.

Abstract—In this letter, we focus on the design of linear and non-linear architectures in amplify-and-forward multiple-input multiple-output orthogonal frequency-division multiplexing relay networks in which different types of services are supported. The goal is to jointly optimize the processing matrices so as to minimize the total power consumption while satisfying the quality-of-service requirements of each service specified as Schur-convex functions of the mean square errors over all assigned subcarriers. It turns out that the optimal solution leads to the diagonalization of the source-relay-destination channel.

Index Terms—MIMO, non-regenerative relay, majorization theory, optimization, additively Schur-convexity, multiplicatively Schur-convexity, quality-of-service requirements, power consumption, multimedia applications.

I. INTRODUCTION

The optimization of linear as well as non-linear architectures for multiple-input multiple-output (MIMO) non-regenerative relay networks has received much attention recently (see for example [1] – [5]). In particular, in [5] the author makes use of majorization theory and propose a unifying framework for minimizing the total power consumption in linear and non-linear multi-hop MIMO relay systems while meeting specific quality-of-service (QoS) requirements given in terms of the mean-square-errors (MSEs) over the different streams. As in [1] – [4], it turns out that the optimal solution leads to the diagonalization of the source-relay-destination channel.

In this work, we extend the results illustrated in [5] to a MIMO orthogonal frequency-division multiplexing (MIMO-OFDM) relay network in which several types of services are supported through spatial multiplexing. Since in practical applications the reliability of each type of transmissions depends on a global performance metric measured over the assigned subcarriers, differently from [5] we reformulate the power minimization problem assuming that the QoS constraint of each service is given as a Schur-convex function of the MSEs over all assigned subcarriers. Interestingly, it turns out that the solution of this problem reduces to the one illustrated in [5] either for a linear or non-linear architecture. The only difference with respect to [5] relies on the structure of the unitary matrix to apply to the transmitted data symbols at the source and destination nodes. This is found to be such that the individual MSEs are all equal to a quantity depending on the specific Schur-convex function. For simplicity, we focus only on a two-hop system in which a single relay is employed. However, all the provided results can be easily extended to other different scenarios in which multi-hops are present. Moreover, it is worth observing that similar conclusions can be achieved if conventional single-hop MIMO-OFDM systems are considered. To the best of our knowledge, this is the first time that the optimization of MIMO-OFDM relay systems with QoS constraints as Schur-convex functions of the MSEs is studied.

II. SYSTEM DESCRIPTION

We consider a MIMO-OFDM relay network in which $N$ subcarriers out of the total number $N_T$ are used to support $K$ different classes of services. The source and destination are equipped with $N_S$ antennas while the relay has $N_R$ antennas$^1$. The $k$th symbol over the $n$th subcarrier is denoted by $s_k(n)$ and is taken from an $L$-ary quadrature amplitude modulation constellation with average power normalized to unity for convenience.

The input data stream is divided into adjacent blocks of $NK \leq \min(N N_R, N N_S)$ symbols, which are transmitted in parallel using the $N$ assigned subcarriers with indices $\{i_n; n = 1, 2, \ldots, N\}$. The vector $\mathbf{s} = [s_1^T, s_2^T, \ldots, s_K^T]^T$ with $s_k = [s_k(1), s_k(2), \ldots, s_k(N)]^T$ is linearly processed by a matrix $\mathbf{U} \in \mathbb{C}^{N N_S \times N K}$ and then launched over the source-relay MIMO channel using $N_S$ OFDM modulators. At the relay, the received signal is processed by a matrix $\mathbf{F} \in \mathbb{C}^{N N_R \times N N_R}$ and forwarded to the destination where the vector $\mathbf{r} \in \mathbb{C}^{N N_R \times 1}$ at the output of the $N_S$ OFDM demodulators takes the form $\mathbf{r} = \mathbf{H U s} + \mathbf{n}$ where $\mathbf{H} = \mathbf{H}_x \mathbf{F H}_y$ is the equivalent channel matrix. In addition, $\mathbf{H}_1 \in \mathbb{C}^{N N_R \times N N_S}$ and $\mathbf{H}_2 \in \mathbb{C}^{N N_R \times N N_R}$ denote the source-relay and relay-destination block diagonal channel matrices given by $\mathbf{H}_1 = \text{blkdiag}(\mathbf{H}_1(1), \mathbf{H}_1(2), \ldots, \mathbf{H}_1(1 N))$ and $\mathbf{H}_2 = \text{blkdiag}(\mathbf{H}_2(1), \mathbf{H}_2(2), \ldots, \mathbf{H}_2(n I))$ with $\mathbf{H}_1(i_n) \in \mathbb{C}^{N R \times N S}$ and $\mathbf{H}_2(i_n) \in \mathbb{C}^{N R \times N R}$ being the channel matrices over the $n$th subcarrier of the corresponding link. In addition, $\mathbf{n} \in \mathbb{C}^{N N_R \times 1}$ is a Gaussian vector with zero mean and covariance matrix $\mathbf{R}_n = \rho_1 \mathbf{H}_2 \mathbf{F F}^H \mathbf{H}_2^H + \rho_2 \mathbf{I}_{N N_R}$ with $\rho_1 > 0$ and $\rho_2 > 0$ being the noise variance over each link. Henceforth, we denote by

$$
\mathbf{H}_1 = \Omega_{H_1} \mathbf{A}^{1/2} \mathbf{V}_H^H \mathbf{H}_1 \quad \text{and} \quad \mathbf{H}_2 = \Omega_{H_2} \mathbf{A}^{1/2} \mathbf{V}_H^H \mathbf{H}_2 \tag{1}
$$

$^1$The following notation is used throughout the paper. Boldface upper and lower-case letters denote matrices and vectors, respectively, while lower-case letters denote scalars. We use $\mathbf{A} = \text{diag}(a_1, a_2, \ldots, a_K)$ to indicate a $K \times K$ diagonal matrix with entries $a_k$ for $k = 1, 2, \ldots, K$ and $\mathbf{A} = \text{blkdiag}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_K)$ to denote a block diagonal matrix. The notations $\mathbf{A}^{-1}$ and $\mathbf{A}^{1/2}$ denote the inverse and square-root of a matrix $\mathbf{A}$. We use $\mathbf{I}_K$ to denote the identity matrix of order $K$ while $[a]_{k, l}$ indicates the $(k, l)$th entry of the enclosed matrix. In addition, we use $\mathbb{E}\{\cdot\}$ for expectation, the superscript $^T$ and $^H$ respectively for transposition and Hermitian transposition.
the singular value decompositions of $\mathbf{H}_1$ and $\mathbf{H}_2$ and assume that the entries of the diagonal matrices $\mathbf{A}_{H_1}$ and $\mathbf{A}_{H_2}$ are in decreasing order.

III. OPTIMIZATION OF THE RELAY NETWORK

The goal of this work is to jointly design the processing matrices so as to minimize the total power consumption $P_T$ given by [5]

$$P_T = \text{tr} \left\{ \mathbf{U} \mathbf{U}^H + \mathbf{F} (\mathbf{H}_1 \mathbf{U} \mathbf{U}^H \mathbf{H}_1^H + \rho_1 \mathbf{I}_M) \mathbf{F}^H \right\}$$

(2)

while satisfying the different QoS requirements of each class of service specified as functions of the MSEs over all subcarriers. To be more specific, the $k$th constraint has the following expression

$$f_k(\{\text{MSE}_k(n)\}_{n=1}^N) \leq \gamma_k \quad \text{for} \quad k = 1, 2, \ldots, K$$

(3)

where $\text{MSE}_k(n)$ denotes the MSE of the $k$th symbol over the $n$th subcarrier while $f_k$ is a generic function that will be specified later. The quantities $\gamma_k$ are design parameters that specify different requirements. Without loss of generality, they are assumed in increasing order, i.e., $\gamma_k \leq \gamma_{k+1}$.

Following [6], we limit our attention to functions $f_k$ that are increasing in each argument and provide closed-form solutions for either additively or multiplicatively Schur-convex functions. This class of functions is of great interest since many different optimization criteria arise in connection with it (see for example [6] for a detailed discussion on the subject).

A. Linear Transceiver Design

When a linear receiver is employed, the vector $\mathbf{r}$ is processed by a matrix $\mathbf{G}$ to obtain $\mathbf{y} = \mathbf{G} \mathbf{H} \mathbf{u} + \mathbf{G} \mathbf{n}$. The MSE matrix $\mathbf{E} = \mathbf{E}(\mathbf{y} - \mathbf{s})(\mathbf{y} - \mathbf{s})^H$ turns out to be given by

$$\mathbf{E} = \mathbf{I}_{KN} + \mathbf{G} (\mathbf{H} \mathbf{U} \mathbf{U}^H \mathbf{H}^H + \mathbf{R}_n) \mathbf{G}^H - \mathbf{G} \mathbf{H} \mathbf{U} \mathbf{U}^H \mathbf{H}^H \mathbf{G}^H$$

(4)

while the $k$th MSE over the $n$th subcarrier is obtained as $\text{MSE}_k(n) = [\mathbf{E}]_{(k-1)N+n, (k-1)N+n}$. For notational convenience, in all subsequent derivations we call $[\mathbf{E}]_{(k-1)N+n, (k-1)N+n} = [\mathbf{E}_k]_{n,n}$ so that we may rewrite $\text{MSE}_k(n) = \{\mathbf{E}_k\}_{n,n}$.

Finding the optimal $\mathbf{G}$ reduces to look for a matrix that satisfies the QoS requirements for any given $\mathbf{U}$ and $\mathbf{F}$. Since $[\mathbf{E}_k]_{n,n}$ is a quadratic function of $\mathbf{G}$, the best we can do is to choose $\mathbf{G}_{\text{opt}}$ so as to minimize each MSE. Indeed, if such a matrix does not satisfy the QoS requirements no other one will [6]. As is well known, this is achieved by choosing $\mathbf{G}_{\text{opt}}$ equal to the Wiener filter. In these circumstances, the MSE matrix in (4) takes the form

$$\mathbf{E} = \mathbf{I}_{KN} - \mathbf{U} \mathbf{U}^H (\mathbf{H} \mathbf{U} \mathbf{U}^H \mathbf{H}^H + \mathbf{R}_n)^{-1} \mathbf{H} \mathbf{U}$$

(5)

Now, we proceed with the design of the matrices $\mathbf{U}$ and $\mathbf{F}$ that solve

$$(P_1): \min_{\mathbf{U}, \mathbf{F}} P_T \quad \text{s.t.} \quad f_k(\{[\mathbf{E}_k]_{n,n}\}_{n=1}^N) \leq \gamma_k \quad \text{for} \quad k = 1, 2, \ldots, K$$

(6)

with $\mathbf{E}$ given by (5). As mentioned before, closed-form solutions for ($\mathbf{U}, \mathbf{F}$) are now computed for $f_k$ being additively Schur-convex. A short list of such functions is given below.

- The arithmetic mean of the MSEs $f_k(\cdot) = \sum_{n=1}^N [\mathbf{E}_k]_{n,n}$.
- The maximum of the MSEs $f_k(\cdot) = \max_{1 \leq n \leq N} [\mathbf{E}_k]_{n,n}$.
- The negative of the minimum of the SINRs $f_k(\cdot) = -\min_{1 \leq n \leq N} \text{SINR}_k(n) = \max_{1 \leq n \leq N} [\mathbf{E}_k]_{n,n}$.

In writing the above list we have used the fact that when the Wiener filter is used at the destination the signal-to-interference noise ratio (SINR) of the $k$th stream over the $n$th subcarrier is given by $\text{SINR}_k(n) = 1/[\mathbf{E}_k]_{n,n} - 1$.

**Proposition 1:** If each $f_k$ is additively Schur-convex, the optimal matrices $\mathbf{U}_{\text{opt}}$ and $\mathbf{F}_{\text{opt}}$ in (6) are given by

$$\mathbf{U}_{\text{opt}} = \tilde{\mathbf{V}}_{H_1} \mathbf{A}^{1/2}_{U} \mathbf{S}^H \quad \mathbf{F}_{\text{opt}} = \tilde{\mathbf{V}}_{H_2} \mathbf{A}^{1/2}_{F} \tilde{\mathbf{Q}}_{H_1}^H$$

(7)

where $\tilde{\mathbf{V}}_{H_1}$, $\tilde{\mathbf{V}}_{H_2}$ and $\tilde{\mathbf{Q}}_{H_1}$ correspond to the $KN$ columns of $\mathbf{V}_{H_1}$, $\mathbf{V}_{H_2}$, and $\mathbf{Q}^{H}_{H_1}$ associated to the $KN$ largest singular values of the corresponding channel matrix while $\mathbf{S} \in \mathbb{C}^{KN \times KN}$ is a suitable unitary matrix such that $[\mathbf{E}_k]_{n,n} = \epsilon_k$ for $n = 1, 2, \ldots, N$ with $\epsilon_k$ obtained as

$$\gamma_k = f_k(\epsilon_k, \epsilon_k, \ldots, \epsilon_k).$$

(9)

In addition, $\mathbf{A}_U = \text{diag}(\lambda_{U,1}, \lambda_{U,2}, \ldots, \lambda_{U,KN})$ and $\mathbf{A}_F = \text{diag}(\lambda_{F,1}, \lambda_{F,2}, \ldots, \lambda_{F,KN})$ with elements in decreasing order.

**Proof:** See Appendix.

As in [5], it follows that $\mathbf{U}_{\text{opt}}$ and $\mathbf{F}_{\text{opt}}$ match the singular vectors of the corresponding channel matrices. Then, the optimal structure of the overall communication system turns out to be diagonal up to a unitary matrix $\mathbf{S}$ that differently from [5] must be chosen so as to guarantee that the diagonal elements of $\mathbf{E}_k$ for $k = 1, 2, \ldots, K$ are all equal to $\epsilon_k$. The latter is always such that $0 < \epsilon_k < 1^2$ and it is computed through (9) on the basis of the given $\gamma_k$ and $f_k$. Assume for example that $f_k$ is the arithmetic mean of the MSEs, then $\epsilon_k$ results given by $\epsilon_k = \gamma_k/N$. On the other hand, $\epsilon_k = \gamma_k$ when $f_k$ takes the maximum of the MSEs over all subcarriers. Once all the quantities $\epsilon_k$ are computed, the unitary matrix $\mathbf{S}$ can be determined using the iterative procedure described in [7].

As shown in [5], the entries of $\mathbf{A}_U$ and $\mathbf{A}_F$ are obtained as the solutions of the following problem:

$$\min_{\{\lambda_{U,i} \geq 0\}, \{\lambda_{F,i} \geq 0\}} \sum_{i=1}^{KN} [\lambda_{U,i} + \lambda_{F,i} (\lambda_{U,i} \lambda_{H,i} + \rho_1)]$$

s.t. $\sum_{i=1}^{j} \lambda_{E,i} \leq \sum_{i=1}^{j} \eta_i$ for $j = 1, 2, \ldots, KN$

where $\eta_i$ is defined as $\eta_i = \epsilon_\nu$ with $\nu \in \{1, 2, \ldots, K\}$ being the integer such that $(\nu-1)N < i \leq \nu N$, while $\lambda_{E,i}$ denotes the $i$th eigenvalue of $\mathbf{E}$. Finding the solution of the above problem is hard since it is not in a convex form. To overcome this difficulty, an upper- and lower-bound to the solution is computed in [5] while an alternative approach with reduced complexity is discussed in [8].

2Observe that $\epsilon_k$ must be larger than zero since a zero MSE can only be achieved when the noise is absent. Vice versa, it must be smaller than 1 whenever we could satisfy the QoS constraint simply neglecting the transmission of the $k$th stream.
B. Non-linear Transceiver Design

When a non-linear receiver with a decision-feedback equalizer is employed at the destination, the vector $z$ at the input of the decision device (assuming correct previous decisions) can be written as $z = (\text{G}H - \text{B})s + \text{Gn}$ where $\text{B} \in \mathbb{C}^{KN \times KN}$ is a strictly upper triangular matrix [4]. The MSE matrix takes the form

$$E = (\text{G}H - C)(\text{G}H - C)^H + \text{GR}_n\text{G}^H$$  \hspace{1cm} (10)

where $C = \text{B} + \text{I}_{KN}$ is a unit-diagonal upper triangular matrix. Using the same arguments adopted for the linear case, the optimal $G$ is easily found to be such that each $[E_k]_{n,n}$ is minimized. This yields [4]

$$G = C(U^H H^H R_n^{-1} H U + I_{KN})^{-1} U^H H^R R_n^{-1}. \hspace{1cm} (11)$$

We substitute (11) into (10) to obtain

$$E = C(U^H H^R R_n^{-1} H U + I_{KN})^{-1} C^H \hspace{1cm} (12)$$

and look for the optimal $C$. As for $G$, the optimal $C$ must be designed so as to minimize each $[E_k]_{n,n}$. Following [4], this is achieved when $C = \text{DL}^H$ where $L$ is the lower triangular matrix obtained from the Cholesky decomposition of $U^H H^R R_n^{-1} H U + I_{KN}$ while the $KN \times KN$ diagonal matrix $D$ is designed such that $[C]_{i,i} = 1$ for $i = 1, 2, \ldots, KN$. Once $C$ has been computed, $B$ is obtained as $B = C - I_{KN}$. Using all the above results, it follows that [5]

$$[E_k]_{n,n} = 1/|L_k|_{n,n}^2 \hspace{1cm} (13)$$

where $[L_k]_{n,n} = [L]_{(k-1)N+n,(k-1)N+n}^H$.

The design of $U$ and $F$ requires to solve (6) with $[E_k]_{n,n}$ given by (13). Closed-form solutions for $U$ and $F$ are now computed for multiplicatively Schur-convex functions. Due to space limitations, we do not report a list of multiplicatively Schur-convex functions and limit to observe that they play the same role as additively Schur-convex functions in the linear case (see [6] for more details).

**Proposition 2:** If each $f_k$ is multiplicatively Schur-convex, then the optimal matrices $U_{opt}$ and $F_{opt}$ are given by

$$U_{opt} = \tilde{V}_H \Lambda_U^{1/2} S^H \hspace{1cm} \text{and} \hspace{1cm} F_{opt} = \tilde{V}_H \Lambda_F^{1/2} \tilde{H}_L^H$$

where $S \in \mathbb{C}^{KN \times KN}$ is unitary and such that

$$[L_k]_{n,n}^{-1} = \sqrt{\epsilon_k} \hspace{.2cm} \text{for} \hspace{.2cm} n = 1, 2, \ldots, N$$  \hspace{1cm} (14)

with $\epsilon_k$ for $k = 1, 2, \ldots, K$ still given by (9). In addition, the matrices $\Lambda_U$ and $\Lambda_F$ are diagonal with elements in decreasing order.

**Proof:** See Appendix.

As for the linear case, it turns out that channel-diagonalizing structure is optimal provided that the symbols are properly rotated by the unitary matrix $S$. The latter must be now chosen such that (14) is satisfied. This can be achieved resorting to the algorithm illustrated in [6].

The entries of $\Lambda_U$ and $\Lambda_F$ are now solutions of the following power allocation problem

$$\min_{\{\lambda_{U,i} \geq 0\}, \{\lambda_{F,i} \geq 0\}} \sum_{i=1}^{KN} [\lambda_{U,i} + \lambda_{F,i} (\lambda_{U,i} \lambda_{H,i} + \rho_1)]$$

subject to

$$\prod_{i=1}^{j} \lambda_{E,i} \leq \prod_{i=1}^{j} \eta_i \hspace{1cm} \text{for} \hspace{1cm} j = 1, 2, \ldots, KN$$

where $\eta_i$ is defined as in Proposition 1. As for the linear case, the above problem is not in a convex form and its solution can be closely approximated resorting to the algorithms discussed in [5] and [8].

IV. NUMERICAL RESULTS

Numerical results are now given to assess the performance of the proposed solutions. The OFDM terminals employ discrete Fourier transform units of size $N_T = 512$ with a cyclic prefix composed of 32 samples and transmit over a bandwidth of 20 MHz. Two different stream are supported over $N = 32$ subcarriers. The number of antennas is $N_S = N_R = 3$. The transmitted symbols belong to a 4-QAM constellation. The channel taps are generated as independent and circularly symmetric Gaussian random variables with zero mean and power delay profile as specified in the ITU IMT-2000 Vehicular-A channel model. The transmit and receive antennas are assumed to be adequately separated so as to make the channel realizations statically independent in the spatial domain.

Fig. 1 illustrates the total power consumption as a function of the QoS constraints when the noise variance over both links is equal and given by 1 or 0.01. For illustrative reasons, the same QoS constraint is imposed for each class of service. This amounts to saying that $\gamma_k = \gamma$ for $k = 1, 2$. Assume for example that $f_k$ is the arithmetic mean of the MSEs then $\epsilon_k = \gamma/N$ for $k = 1, 2$. On the other hand, if $f_k$ is the maximum MSE then $\epsilon_k = \gamma$ for $k = 1, 2$. The curves labelled with RC-L and RC-NL refer respectively to a system in which a linear or a nonlinear receiver is employed in conjunction with the reduced-complexity power allocation algorithm proposed in [8]. As expected, the results of Fig. 1 indicate that a non-linear architecture provides the best performance for all the investigated values of $\gamma$. 

![Fig. 1. Total power consumption when equal QoS constraints are given with $N = 32$, $N_S = N_R = 3$, $K = 2$](image-url)
V. CONCLUSIONS

We have discussed the optimization of linear and non-linear architectures for MIMO-OFDM relay networks to minimize the total power consumption while satisfying QoS requirements given as additively/multiplicatively Schur-convex functions of the MSEs of each stream over all subcarriers. Interestingly, it is found that for both classes of functions the diagonalizing structure is optimal provided that the transmitted data symbols are properly rotated before channel diagonalization.

APPENDIX

The proof of Proposition 1 relies on showing that if each $f_k$ is additively Schur-convex then the original problem ($P_1$) in (6) is equivalent to the following one ($P_2$):

$\min_{U,F} P_T \quad \text{s.t.} \quad [E_k]_{1,1} = \cdots = [E_k]_{N,N} \leq \epsilon_k \quad \forall k$

where $P_T$ is given by (2) and $\epsilon_k$ is such that

$$f_k(1_{\epsilon_k}) = \gamma_k$$

(15)

with $1_{\epsilon_k}$ being the $N-$dimensional vector defined as $1_k = [\epsilon_k, \epsilon_k, \ldots, \epsilon_k]^T$. The above problem is formally equivalent to the one discussed in [5] meaning that the matrices $U$ and $F$ solving ($P_2$) have the same form of those computed in [5] and are given by (7) in the text.

For notational convenience, we denote by $P_T(U,F)$ the transmit power required by the matrices $(U,F)$ and call $[E_k(U,F)]_{n,n}$ the corresponding MSE of the $k$th symbol over the $n$th subcarrier.

To establish the equivalence of ($P_1$) and ($P_2$), it is enough to show that for any pair $(U_1,F_1)$ in the feasible set of ($P_1$) it is always possible to find a corresponding pair $(U_2,F_2)$ in the feasible set of ($P_2$) for which the same transmit power is required, i.e., $P_T(U_1,F_1) = P_T(U_2,F_2)$ and vice-versa. We start assuming that $(U_1,F_1)$ is in the feasible set of ($P_1$), i.e.,

$$f_k([E_k(U_1,F_1)]_{n,n}) = 1 \leq \gamma_k$$

(16)

Using the results illustrated [7], it can be shown that there always exists a unitary matrix $S$ such that the MSEs become all equal to their arithmetic mean, i.e.,

$$[E_k(U_1S,F_1)]_{n,n} = \frac{1}{N} \sum_{j=1}^{N} [E_k(U_1,F_1)]_{j,j} = \theta_k.$$

To proceed further, denote by $e_k(U_1,F_1)$ the vector collecting the MSEs of the $k$th stream, i.e., $e_k(U_1,F_1) = ([E_k(U_1,F_1)]_{1,1}, [E_k(U_1,F_1)]_{2,2}, \ldots, [E_k(U_1,F_1)]_{N,N})^T$. From [9], it is seen that $1_{\theta_k} \preceq e_k(U_1,F_1)$ where $1_{\theta_k}$ is the $N-$dimensional vector defined as $1_{\theta_k} = [\theta_k, \theta_k, \ldots, \theta_k]^T$. If $f_k$ is additively Schur-convex then $f_k(1_{\theta_k}) \leq f_k(e_k(U_1,F_1))$. From which using (16) it follows that $f_k(1_{\theta_k}) \leq \gamma_k$ or, equivalently, $f_k(1_{\theta_k}) \leq f_k(1_{\epsilon_k})$ where we have used the definition in (15). Since $f_k$ is a non-decreasing function of its arguments, from $f_k(1_{\theta_k}) \leq f_k(1_{\epsilon_k})$ it follows that

$$[E_k(U_1S,F_1)]_{n,n} = \theta_k \leq \epsilon_k$$

which amounts to saying that $(U_1S,F_1)$ is in the feasible set of ($P_2$). In addition, from (2) it easily follows that $P_T(U_1,F_1) = P_T(U_1S,F_1)$. Then, we may conclude that for any feasible $(U_1,F_1)$ in ($P_1$) there always exists a pair $(U_2,F_2)$ of the form $(U_2,F_2) = (U_1S,F_1)$, which is in the feasible set of ($P_2$) and requires the same amount of transmit power.

We now prove the reverse part. Let $(U_2,F_2)$ be in the feasible set of ($P_2$), i.e.,

$$[E_k(U_2,F_2)]_{n,n} = \epsilon_k \quad \forall k$$

with required transmit power $P_T(U_2,F_2)$. Letting

$$[E_k(U_2,F_2)]_{n,n} = \theta_k \quad \forall n$$

and exploiting the fact that $f_k$ is a non-decreasing function of its arguments, using (15) and (17) we may write

$$f_k([E_k(U_2,F_2)]_{n,n}) = f_k(1_{\theta_k}) \leq f_k(1_{\epsilon_k}) = \gamma_k$$

from which it follows that $(U_2,F_2)$ is in the feasible set of ($P_1$). Therefore, setting $(U_1,F_1) = (U_2,F_2)$ yields the desired result. This completes the proof of Proposition 1.

The proof of Proposition 2 is much similar to that of Proposition 1. For this reason, in the sequel we report only the major differences. The first part relies on the observation that it is always possible to find a unitary matrix $P$ such that the MSEs given by (13) become equal to their geometric mean [6], i.e.,

$$[E_k(U_1P,F_1)]_{n,n} = \left( \prod_{j=1}^{N} [E_k(U_1,F_1)]_{j,j} \right)^{\frac{1}{N}} = \theta_n.$$

In addition, if $f_k$ is multiplicatively Schur-convex then $f_k(1_{\theta_k}) \leq f_k(e_k(U_1,F_1))$ from which using the same arguments of before it easily follows that $(U_1P,F_1)$ is in the feasible set of ($P_2$) and requires the same amount of power. The reverse part is straightforward.

REFERENCES