



Optimum Coherent Detection in Gaussian Disturbance

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Binary Hypothesis Testing Problem

■ The function of a surveillance radar is to ascertain whether targets are present in the data. Given the observed **vector \mathbf{z}** , the signal processor must make a decision as to which of the two hypotheses is true:

$$\begin{cases} \mathbf{z} = \mathbf{d} & H_0 : \text{Target absent} \\ \mathbf{z} = \mathbf{s}_t + \mathbf{d} & H_1 : \text{Target present} \end{cases}$$

- The test is implemented on-line for each **range cell under test (CUT)**
 - The detection is a **binary hypothesis problem** whereby we decide either hypothesis H_0 is true and only disturbance is present or hypothesis H_1 is true and a target signal is present with disturbance.
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Neyman-Pearson Criterion

- According to the **Neyman-Pearson (NP) criterion** (maximize P_D while keeping constant P_{FA}), the optimal decision strategy is a **likelihood ratio test (LRT)**:

$$\Lambda(\mathbf{z}) = \frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} \underset{H_0}{>} \underset{H_1}{<} e^\eta \quad \Rightarrow \quad \ln \Lambda(\mathbf{z}) = \ln \left(\frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} \right) \underset{H_0}{>} \underset{H_1}{<} \eta$$

$p_{\mathbf{z}|H_i}(\mathbf{z}|H_i)$ is the probability density function (PDF) of the random vector \mathbf{z} under the hypothesis H_i , $i=0,1$.

η is the detection threshold (set according to the desired P_{FA}).

The Optimal Detector

- The complex multidimensional PDF of Gaussian disturbance is given by:

$$p_{\mathbf{z}|H_0}(\mathbf{z}|H_0) = p_{\mathbf{d}}(\mathbf{z}) = \frac{1}{\pi^N |\mathbf{R}|} \exp(-\mathbf{z}^H \mathbf{R}^{-1} \mathbf{z}) \quad \mathbf{z}|H_0 \in \mathcal{CN}(\mathbf{0}, \mathbf{R})$$

$N \times 1$ data vector

$|\mathbf{R}| = \det\{\mathbf{R}\}$, $\mathbf{R} = \mathbf{R}^H \rightarrow$ Hermitian matrix

- The complex multidimensional PDF of *target + disturbance*: $p_{\mathbf{z}|H_1}(\mathbf{z}|H_1) = ?$
- It depends on the target signal model: $\mathbf{z} = \mathbf{s}_t + \mathbf{d}$

- If the target vector is deterministic:

$$p_{\mathbf{z}|H_1}(\mathbf{z}|H_1) = p_{\mathbf{z}|H_0}(\mathbf{z} - \mathbf{s}_t|H_0)$$

The Optimal Detector

- If the target vector is random with known PDF:

$$p_{\mathbf{z}|H_1}(\mathbf{z}|H_1) = E_{\mathbf{s}_t} \left\{ p_{\mathbf{z}|H_0}(\mathbf{z} - \mathbf{s}_t | H_0) \right\} = \int \underbrace{\frac{1}{\pi^N |\mathbf{R}|} \exp\left(-(\mathbf{z} - \mathbf{s}_t)^H \mathbf{R}^{-1} (\mathbf{z} - \mathbf{s}_t)\right)}_{p_{\mathbf{z}|H_0}(\mathbf{z} - \mathbf{s}_t | H_0)} \cdot p_{\mathbf{s}_t}(\mathbf{s}_t) d\mathbf{s}_t$$

- Under some assumptions, we have found for the target signal:

$$\mathbf{s}_t = \beta \mathbf{p}(f_d) \quad [\beta \text{ is the target complex amplitude}]$$

$N \times 1$

$\mathbf{p}(f_d)$ **Steering Vector**

$N \times 1$

$$f_d = \frac{2v_r T_r}{\lambda} \quad \text{Doppler frequency} \Leftrightarrow \text{target radial velocity}$$

Narrowband Slowly-Fluctuating Target Signal Model

$$\mathbf{p}(f_d) = \begin{bmatrix} 1 \\ e^{j2\pi f_d} \\ \vdots \\ e^{j2\pi(N-1)f_d} \end{bmatrix}$$

$N \times 1$

■ Temporal Steering Vector

It has the Vandermonde form because the waveform PRF is uniform and target velocity is constant during the CPI.

- The detection algorithm is optimized for a specific Doppler.
 - Since the target velocity is unknown a-priori, \mathbf{p} is a known function of unknown parameters, so the radar receiver should implement multiple detectors that form a **filter bank** to cover all potential target Doppler frequencies.
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Target Signal Models

$$\mathbf{s}_t = \beta \mathbf{p}(f_d)$$

$N \times 1$

■ Different models of \mathbf{s}_t have been investigated to take into account different degrees of *a priori* knowledge on the target signal:

(1) \mathbf{s}_t perfectly known

(2) $\mathbf{s}_t = \beta \mathbf{p}$ with $\beta \in \mathcal{CN}(0, \sigma_s^2)$ i.e., Swerling I model, and \mathbf{p} perfectly known;

(3) $\mathbf{s}_t = \beta \mathbf{p}$ with $|\beta|$ deterministic and $\angle \beta$ random, uniformly distributed in $[0, 2\pi)$,
i.e. Swerling 0 (or Swerling V) model, and \mathbf{p} perfectly known

(4) $\mathbf{s}_t = \beta \mathbf{p}$ with β unknown deterministic and \mathbf{p} perfectly known;

Perfectly Known Target Signal - Case #1

- The optimal NP decision strategy is a LRT (or log-LRT):

$$\begin{aligned}
 l(\mathbf{z}) = \ln \Lambda(\mathbf{z}) &= \ln \frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = \mathbf{z}^H \mathbf{R}^{-1} \mathbf{z} - (\mathbf{z} - \mathbf{s}_t)^H \mathbf{R}^{-1} (\mathbf{z} - \mathbf{s}_t) \\
 &= \mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{z} + \mathbf{z}^H \mathbf{R}^{-1} \mathbf{s}_t - \mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t = \underbrace{2\Re\{ \mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{z} \}}_{\substack{H_1 \\ > \\ H_0}} - \mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t > \eta
 \end{aligned}$$

- It is the so-called **coherent whitening matched filter (CWMF)** detector:

$$\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{z} = \mathbf{s}_t^H \mathbf{R}^{-1/2} \mathbf{R}^{-1/2} \mathbf{z} = \mathbf{s}_t^H \left(\mathbf{R}^{-1/2} \right)^H \mathbf{R}^{-1/2} \mathbf{z} = \left(\mathbf{R}^{-1/2} \mathbf{s}_t \right)^H \mathbf{R}^{-1/2} \mathbf{z} = \bar{\mathbf{s}}_t^H \bar{\mathbf{z}}$$

$\bar{\mathbf{s}}_t \triangleq \mathbf{R}^{-1/2} \mathbf{s}_t$, $\bar{\mathbf{z}} \triangleq \mathbf{R}^{-1/2} \mathbf{z}$ whitening transformation matched filtering

$$E\{\bar{\mathbf{z}}\bar{\mathbf{z}}^H\} = E\{\mathbf{R}^{-1/2} \mathbf{z} \mathbf{z}^H \mathbf{R}^{-1/2}\} = \mathbf{R}^{-1/2} E\{\mathbf{z} \mathbf{z}^H\} \mathbf{R}^{-1/2} = \mathbf{R}^{-1/2} \mathbf{R} \mathbf{R}^{-1/2} = \mathbf{I}$$

Performance of the coherent WMF

$$l(\mathbf{z}) = 2\Re e \left\{ \mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{z} \right\} \underset{H_0}{\overset{H_1}{>}} \eta$$

$\mathbf{z}|H_0 \in \mathcal{CN}(0, \mathbf{R})$, $\mathbf{z}|H_1 \in \mathcal{CN}(\mathbf{s}_t, \mathbf{R}) \Rightarrow l$ is a real Gaussian r.v.

$$m_0 = E\{l|H_0\} = 2\Re e \left\{ \mathbf{s}_t^H \mathbf{R}^{-1} E\{\mathbf{z}|H_0\} \right\} = 0$$

$$m_1 = E\{l|H_1\} = 2\Re e \left\{ \mathbf{s}_t^H \mathbf{R}^{-1} E\{\mathbf{z}|H_1\} \right\} = 2\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t$$

- The fact that \mathbf{d} is a complex circular Gaussian random vector implies that [Ch.13, Kay98]:

$$E\{\mathbf{d}\mathbf{d}^H\} = \mathbf{R} = \mathbf{R}^H \quad \text{and} \quad E\{\mathbf{d}\mathbf{d}^T\} = 0$$

Performance of the coherent WMF

$$\begin{aligned}
 \sigma_0^2 &= \text{var}\{l|H_0\} = E\left\{\left|l - E\{l|H_0\}\right|^2 | H_0\right\} \\
 &= E\left\{\left|2\Re\{s_t^H \mathbf{R}^{-1} \mathbf{z}\}\right|^2 | H_0\right\} = E\left\{\left|s_t^H \mathbf{R}^{-1} \mathbf{z} + \mathbf{z}^H \mathbf{R}^{-1} s_t\right|^2 | H_0\right\} \\
 &= E\left\{\left(s_t^H \mathbf{R}^{-1} \mathbf{d} + \mathbf{d}^H \mathbf{R}^{-1} s_t\right)\left(s_t^H \mathbf{R}^{-1} \mathbf{d} + \mathbf{d}^H \mathbf{R}^{-1} s_t\right)^H\right\} \\
 &= E\left\{s_t^H \mathbf{R}^{-1} \mathbf{d} \mathbf{d}^H \mathbf{R}^{-1} s_t + s_t^H \mathbf{R}^{-1} \mathbf{d} s_t^H \mathbf{R}^{-1} \mathbf{d} + \mathbf{d}^H \mathbf{R}^{-1} s_t \mathbf{d}^H \mathbf{R}^{-1} s_t + \mathbf{d}^H \mathbf{R}^{-1} s_t s_t^H \mathbf{R}^{-1} \mathbf{d}\right\} \\
 &= E\left\{s_t^H \mathbf{R}^{-1} \mathbf{d} \mathbf{d}^T \left(\mathbf{R}^{-1}\right)^* s_t^* + 2s_t^H \mathbf{R}^{-1} \mathbf{d} \mathbf{d}^H \mathbf{R}^{-1} s_t + s_t^T \left(\mathbf{R}^{-1}\right)^* \mathbf{d}^* \mathbf{d}^H \mathbf{R}^{-1} s_t\right\} \\
 &= E\left\{2s_t^H \mathbf{R}^{-1} \mathbf{d} \mathbf{d}^T \left(\mathbf{R}^{-1}\right)^* s_t^* + 2s_t^H \mathbf{R}^{-1} \mathbf{d} \mathbf{d}^H \mathbf{R}^{-1} s_t\right\} \\
 &= 2s_t^H \mathbf{R}^{-1} E\{\mathbf{d} \mathbf{d}^T\} \mathbf{R}^{-1} s_t^* + 2s_t^H \mathbf{R}^{-1} E\{\mathbf{d} \mathbf{d}^H\} \mathbf{R}^{-1} s_t \\
 &= 2s_t^H \mathbf{R}^{-1} \mathbf{R} \mathbf{R}^{-1} s_t = 2s_t^H \mathbf{R}^{-1} s_t
 \end{aligned}$$

$$\sigma_1^2 = \text{var}\{l|H_1\} = E\left\{\left|l - E\{l|H_1\}\right|^2 | H_1\right\} = 2s_t^H \mathbf{R}^{-1} s_t$$

we used the fact that $s_t^H \mathbf{R}^{-1} \mathbf{d} = \left(s_t^H \mathbf{R}^{-1} \mathbf{d}\right)^T = \mathbf{d}^T \mathbf{R}^{-T} s_t^* = \mathbf{d}^T \left(\mathbf{R}^{-1}\right)^* s_t^*$ and $E\{\mathbf{d} \mathbf{d}^T\} = 0$

Performance of the coherent WMF

- The probability of detection (P_D) and probability of false alarm (P_{FA}) can be calculated using the statistics we have just derived:

$$\begin{aligned} P_{FA} &= \Pr\{l > \eta | H_0\} = \int_{\eta}^{+\infty} p_{l|H_0}(l|H_0) dl = \int_{\eta}^{+\infty} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{(l-m_0)^2}{2\sigma_0^2}\right) dl \\ &= \int_{\frac{\eta-m_0}{\sigma_0}}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx = Q\left(\frac{\eta-m_0}{\sigma_0}\right) = Q\left(\frac{\eta}{\sqrt{2\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t}}\right) \end{aligned}$$

$$P_D = \Pr\{l > \eta | H_1\} = \int_{\eta}^{+\infty} p_{l|H_1}(l|H_1) dl = Q\left(\frac{\eta-m_1}{\sigma_1}\right) = Q\left(\frac{\eta - 2\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t}{\sqrt{2\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t}}\right)$$

What is the meaning of $2\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t$?

Performance of the coherent WMF

■ The filter output can be written as : $y = \mathbf{w}^H \mathbf{z} = \mathbf{w}^H (\mathbf{s}_t + \mathbf{d}) = \mathbf{w}^H \mathbf{s}_t + \mathbf{w}^H \mathbf{d}$ (H_1)

■ The **Signal-to-Noise ratio (SNR)** at the output is:

$$SNR = \frac{E\left\{|\mathbf{w}^H \mathbf{s}_t|^2\right\}}{E\left\{|\mathbf{w}^H \mathbf{d}|^2\right\}} = \frac{E\left\{\mathbf{w}^H \mathbf{s}_t \mathbf{s}_t^H \mathbf{w}\right\}}{E\left\{\mathbf{w}^H \mathbf{d} \mathbf{d}^H \mathbf{w}\right\}} = \frac{\mathbf{w}^H E\left\{\mathbf{s}_t \mathbf{s}_t^H\right\} \mathbf{w}}{\mathbf{w}^H E\left\{\mathbf{d} \mathbf{d}^H\right\} \mathbf{w}} = \frac{\mathbf{w}^H \mathbf{s}_t \mathbf{s}_t^H \mathbf{w}}{\mathbf{w}^H \mathbf{R} \mathbf{w}} = \frac{|\mathbf{w}^H \mathbf{s}_t|^2}{\mathbf{w}^H \mathbf{R} \mathbf{w}}$$

if the target signal is deterministic

■ Let's apply this consideration to our case where $\mathbf{w} = \mathbf{R}^{-1} \mathbf{s}_t$ and

$$\Re\left\{\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{z}\right\} = \Re\left\{\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t\right\} + \Re\left\{\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{d}\right\} = \mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t + \Re\left\{\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{d}\right\}$$

Performance of the coherent WMF

- Therefore, the SINR at the output of the **coherent WMF** (CWMF) is:

$$SNR_{CWMF} = \frac{(\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t)^2}{E \left\{ \left(\Re \{ \mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{d} \} \right)^2 \right\}} = \frac{(\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t)^2}{E \left\{ \left(\frac{\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{d} + \mathbf{d}^H \mathbf{R}^{-1} \mathbf{s}_t}{2} \right)^2 \right\}}$$

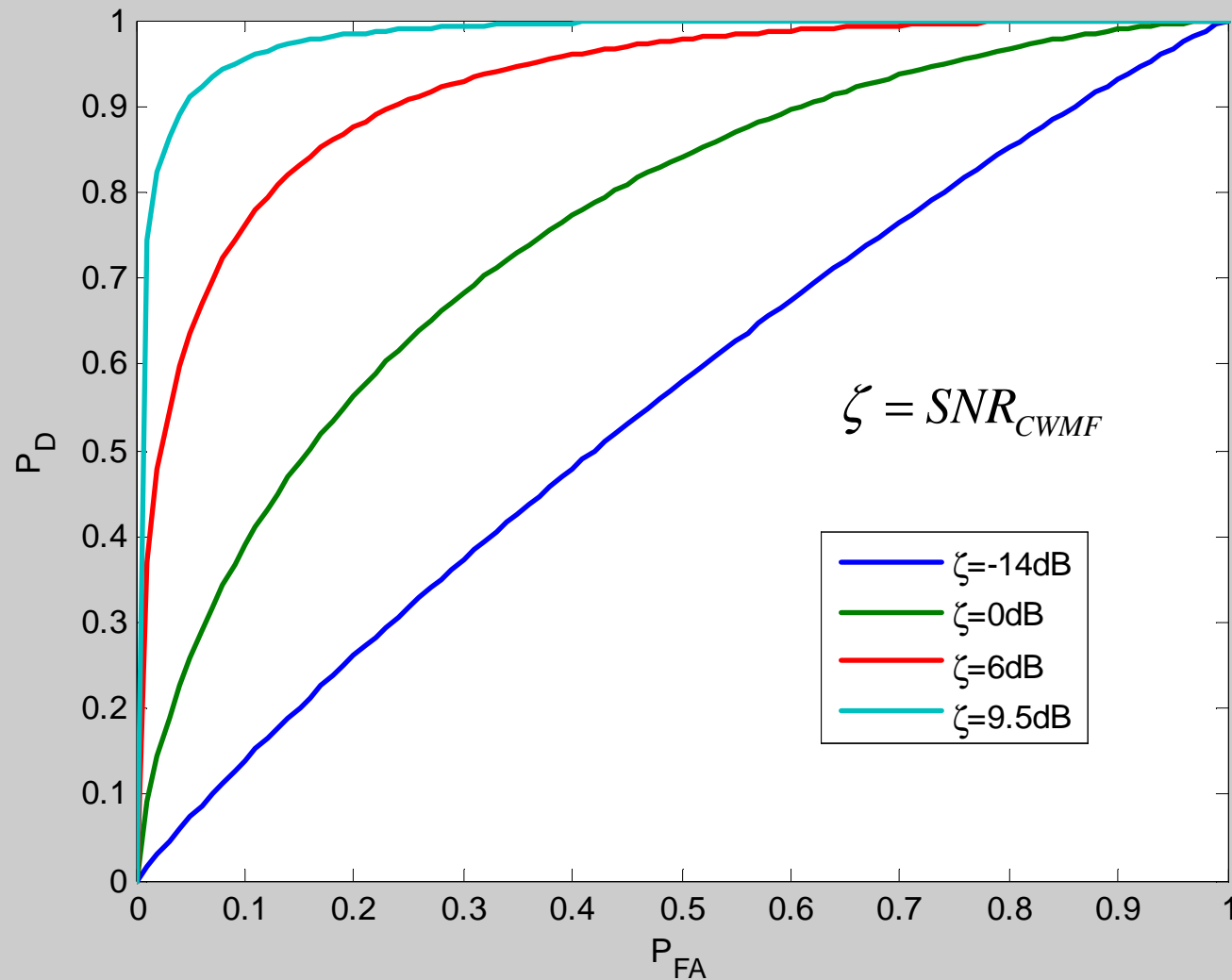
$$= \frac{4(\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t)^2}{2\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t} = 2\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t$$

$$P_{FA} = Q \left(\frac{\eta}{\sqrt{SNR_{CWMF}}} \right) \Rightarrow \eta = \sqrt{SNR_{CWMF}} \cdot Q^{-1}(P_{FA})$$

$$P_D = Q \left(\frac{\eta - SNR_{CWMF}}{\sqrt{SNR_{CWMF}}} \right) \Rightarrow P_D = Q \left(Q^{-1}(P_{FA}) - \sqrt{SNR_{CWMF}} \right)$$

Performance of the coherent WMF

Receiver Operating Characteristic (ROC) curves:



- The ROC is a function of only the SNR and the threshold.
- Here, the threshold is varied to cover the entire range of interest and the ROC curve is plotted for four different SNR values.

Swerling I Target Signal - Case #2

$$\mathbf{s}_t = \beta \mathbf{p}(f_d), \quad \beta \in \mathcal{CN}(0, \sigma_s^2), \quad \mathbf{p} \text{ perfectly known}$$

$$\beta = |\beta| e^{j\varphi} \in \mathcal{CN}(0, \sigma_s^2) \Rightarrow |\beta| \text{ is Rayleigh distributed, } E\{|\beta|^2\} = \sigma_s^2$$

φ is uniformly distributed on $[0, 2\pi)$

- Based on these assumptions:

$$\mathbf{z}|H_0 \in \mathcal{CN}(0, \mathbf{R}), \quad \mathbf{z}|H_1 \in \mathcal{CN}(0, \sigma_s^2 \mathbf{p}\mathbf{p}^H + \mathbf{R})$$

$$E\{\mathbf{z}\mathbf{z}^H | H_1\} = E\{(\beta \mathbf{p} + \mathbf{d})(\beta \mathbf{p} + \mathbf{d})^H\} = E\{|\beta|^2\} \mathbf{p}\mathbf{p}^H + \mathbf{R} = \sigma_s^2 \mathbf{p}\mathbf{p}^H + \mathbf{R}$$

Swerling I Target Signal - Case #2

$$p_{\mathbf{z}|H_0}(\mathbf{z}|H_0) = \frac{1}{\pi^N |\mathbf{R}|} \exp(-\mathbf{z}^H \mathbf{R}^{-1} \mathbf{z})$$

$$p_{\mathbf{z}|H_1}(\mathbf{z}|H_1) = \frac{1}{\pi^N |\sigma_s^2 \mathbf{p} \mathbf{p}^H + \mathbf{R}|} \exp(-\mathbf{z}^H (\sigma_s^2 \mathbf{p} \mathbf{p}^H + \mathbf{R})^{-1} \mathbf{z})$$

- The log-likelihood ratio (log-LR or LLR) is given by:

$$\ln \frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = \ln |\mathbf{R}| - \ln |\sigma_s^2 \mathbf{p} \mathbf{p}^H + \mathbf{R}| + \mathbf{z}^H \mathbf{R}^{-1} \mathbf{z} - \mathbf{z}^H (\sigma_s^2 \mathbf{p} \mathbf{p}^H + \mathbf{R})^{-1} \mathbf{z}$$

- By making use of *Woodbury's identity*:

$$(\sigma_s^2 \mathbf{p} \mathbf{p}^H + \mathbf{R})^{-1} = \mathbf{R}^{-1} - \frac{\sigma_s^2 \mathbf{R}^{-1} \mathbf{p} \mathbf{p}^H \mathbf{R}^{-1}}{1 + \sigma_s^2 \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}}$$

Swerling I Target Signal - Case #2

$$\ln \Lambda(\mathbf{z}) = \ln |\mathbf{R}| - \ln \left| \sigma_s^2 \mathbf{p} \mathbf{p}^H + \mathbf{R} \right| + \sigma_s^2 \frac{\mathbf{z}^H \mathbf{R}^{-1} \mathbf{p} \mathbf{p}^H \mathbf{R}^{-1} \mathbf{z}}{1 + \sigma_s^2 \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}} \underset{H_0}{\overset{H_1}{>}} \eta$$

- By incorporating the non-essential terms in the threshold, we derive that in this case the optimal NP decision strategy is:

$$\left| \mathbf{p}^H \mathbf{R}^{-1} \mathbf{z} \right|^2 \underset{H_0}{\overset{H_1}{>}} \eta$$

- Again, the optimal decision strategy requires calculation of the WMF output, but instead of taking the real part of the output we calculate the modulo of the output: this is due to the fact that the target phase in this case is unknown.
 - This is sometimes called the **noncoherent WMF**
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Performance of the non-coherent WMF

- Performance of the non-coherent WMF: **Signal-to-Noise power ratio (SNR)**

$$|\mathbf{w}^H \mathbf{z}|^2 = |\mathbf{p}^H \mathbf{R}^{-1} \mathbf{z}|^2 \begin{matrix} > & H_1 \\ < & H_0 \end{matrix} \eta$$

$\mathbf{w} = \kappa \mathbf{R}^{-1} \mathbf{p}$ optimal weights

$$SNR = \frac{E\left\{|\mathbf{w}^H \mathbf{s}_t|^2\right\}}{E\left\{|\mathbf{w}^H \mathbf{d}|^2\right\}} = \frac{E\left\{|\mathbf{w}^H \mathbf{s}_t|^2\right\}}{\mathbf{w}^H \mathbf{R} \mathbf{w}} \Rightarrow SNR_{WMF} = E\left\{|\beta|^2\right\} \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}$$

$$SNR_{WMF} = \sigma_s^2 \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}, \quad \beta \in \mathcal{CN}(0, \sigma_s^2)$$

Performance of the non-coherent WMF – SW1

- Performance of the non-coherent WMF when the target complex amplitude is a zero-mean circular complex Gaussian r.v. (Swerling I).


$$\mathbf{z}|H_0 \in \mathcal{CN}(0, \mathbf{R}), \quad \mathbf{z}|H_1 \in \mathcal{CN}(0, \sigma_s^2 \mathbf{p}\mathbf{p}^H + \mathbf{R})$$

$$\Rightarrow Y \triangleq \mathbf{p}^H \mathbf{R}^{-1} \mathbf{z} \text{ is a complex circular Gaussian r.v. } |H_i$$

$$\Rightarrow Y|H_i \in \mathcal{CN}(m_i, \sigma_i^2), i = 0, 1$$

$$m_0 = E\{Y|H_0\} = \mathbf{p}^H \mathbf{R}^{-1} E\{\mathbf{z}|H_0\} = \mathbf{p}^H \mathbf{R}^{-1} E\{\mathbf{d}\} = 0$$

$$m_1 = E\{Y|H_1\} = \mathbf{p}^H \mathbf{R}^{-1} E\{\mathbf{z}|H_1\} = \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p} \cdot E\{\beta\} + \mathbf{p}^H \mathbf{R}^{-1} E\{\mathbf{d}\} = 0$$


$$X = |Y|^2 = Y_R^2 + Y_I^2 \Rightarrow X|H_i \in \text{Exp}(\sigma_i^2), i = 0, 1$$

$$p_{X|H_i}(x|H_i) = \frac{1}{\sigma_i^2} e^{-\frac{x}{\sigma_i^2}} u(x)$$

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

Performance of the non-coherent WMF – SW1

$$\begin{aligned}\sigma_0^2 &= \text{var}\{Y|H_0\} = E\{|Y|^2|H_0\} = E\{|\mathbf{p}^H \mathbf{R}^{-1} \mathbf{z}|^2|H_0\} = E\{|\mathbf{p}^H \mathbf{R}^{-1} \mathbf{d}|^2\} \\ &= E\{\mathbf{p}^H \mathbf{R}^{-1} \mathbf{d} \mathbf{d}^H \mathbf{R}^{-1} \mathbf{p}\} = \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}\end{aligned}$$

$$\begin{aligned}\sigma_1^2 &= \text{var}\{Y|H_1\} = E\{|Y|^2|H_1\} = E\{|\mathbf{p}^H \mathbf{R}^{-1} \mathbf{z}|^2|H_1\} \\ &= E\{|\beta \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p} + \mathbf{p}^H \mathbf{R}^{-1} \mathbf{d}|^2\} = E\{|\beta \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}|^2 + |\mathbf{p}^H \mathbf{R}^{-1} \mathbf{d}|^2\} \\ &= \sigma_s^2 (\mathbf{p}^H \mathbf{R}^{-1} \mathbf{p})^2 + \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p} = (\mathbf{p}^H \mathbf{R}^{-1} \mathbf{p})(1 + \sigma_s^2 \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p})\end{aligned}$$

$$P_{FA} = \int_{\eta}^{+\infty} p_{X|H_0}(x|H_0) dx = e^{-\frac{\eta}{\sigma_0^2}}$$

$$P_D = \int_{\eta}^{+\infty} p_{X|H_1}(x|H_1) dx = e^{-\frac{\eta}{\sigma_1^2}}$$

Performance of the non-coherent WMF – SW1

$$P_{FA} = e^{-\frac{\eta}{\sigma_0^2}} \Rightarrow \eta = \sigma_0^2 \ln(1/P_{FA})$$

$$P_D = e^{-\frac{\eta}{\sigma_1^2}} = e^{-\frac{\sigma_0^2 \ln(1/P_{FA})}{\sigma_1^2}} = \left(e^{-\ln(1/P_{FA})} \right)^{\frac{\sigma_0^2}{\sigma_1^2}} = (P_{FA})^{\frac{\sigma_0^2}{\sigma_1^2}}$$

where $\frac{\sigma_1^2}{\sigma_0^2} = 1 + \sigma_s^2 \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p} = 1 + SNR_{WMF}$

- Receiver Operating Characteristic (ROC) curves:

$$P_D = (P_{FA})^{\frac{1}{1+SNR_{WMF}}}$$

- P_D is a monotonic increasing function of SNR_{WMF} .
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Swerling 0 Target Signal – Case #3

- If the amplitude is non fluctuating and the phase is uniformly distributed:

$$\beta = |\beta|e^{j\varphi} \Rightarrow |\beta| \text{ is deterministic (nonfluctuating)}$$
$$\varphi \text{ is uniformly distributed on } [0, 2\pi)$$
$$\mathbf{z}|\varphi, H_1 \in \mathcal{CN}(|\beta|e^{j\varphi}\mathbf{p}, \mathbf{R})$$

$$\Lambda(\mathbf{z}) = \frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = \frac{\int_{-\infty}^{+\infty} p_{\mathbf{z},\varphi|H_1}(\mathbf{z}, \varphi|H_1)d\varphi}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = \frac{\int_{-\infty}^{+\infty} p_{\mathbf{z}|\varphi, H_1}(\mathbf{z}|\varphi, H_1)p_{\varphi|H_1}(\varphi|H_1)d\varphi}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)}$$
$$= \frac{\int_{-\infty}^{+\infty} p_{\mathbf{z}|\varphi, H_1}(\mathbf{z}|\varphi, H_1)p_{\varphi}(\varphi)d\varphi}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = \frac{1}{2\pi} \int_0^{2\pi} p_{\mathbf{z}|\varphi, H_1}(\mathbf{z}|\varphi, H_1)d\varphi}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)}$$

Swerling 0 Target Signal

$$\begin{aligned}
 \Lambda(\mathbf{z}) &= \frac{\frac{1}{2\pi} \int_0^{2\pi} p_{\mathbf{z}|\varphi, H_1}(\mathbf{z}|\varphi, H_1) d\varphi}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} \\
 &= \frac{\frac{1}{2\pi} \int_0^{2\pi} \exp\left(-(\mathbf{z} - |\beta|e^{j\varphi}\mathbf{p})^H \mathbf{R}^{-1}(\mathbf{z} - |\beta|e^{j\varphi}\mathbf{p})\right) d\varphi}{\exp\left(-\mathbf{z}^H \mathbf{R}^{-1}\mathbf{z}\right)} \\
 &= \frac{1}{2\pi} \int_0^{2\pi} \exp\left(|\beta|e^{-j\varphi}\mathbf{p}^H \mathbf{R}^{-1}\mathbf{z} + |\beta|e^{j\varphi}\mathbf{z}^H \mathbf{R}^{-1}\mathbf{p} - |\beta|^2 \mathbf{p}^H \mathbf{R}^{-1}\mathbf{p}\right) d\varphi \\
 &= \frac{1}{2\pi} \int_0^{2\pi} \exp\left(2|\beta|\left|\mathbf{p}^H \mathbf{R}^{-1}\mathbf{z}\right| \cos(\varphi - \varphi_0) - |\beta|^2 \mathbf{p}^H \mathbf{R}^{-1}\mathbf{p}\right) d\varphi \\
 &= \frac{1}{2\pi} \int_0^{2\pi} \exp\left(2|\beta|\left|\mathbf{p}^H \mathbf{R}^{-1}\mathbf{z}\right| \cos(\varphi - \varphi_0)\right) d\varphi \cdot e^{-|\beta|^2 \mathbf{p}^H \mathbf{R}^{-1}\mathbf{p}},
 \end{aligned}$$

where $\varphi_0 \triangleq \angle \mathbf{p}^H \mathbf{R}^{-1}\mathbf{z}$

Swerling 0 Target Signal – Case #3

$$\begin{aligned}\Lambda(\mathbf{z}) &= \frac{1}{2\pi} \int_0^{2\pi} \exp\left(2|\beta| \left|\mathbf{p}^H \mathbf{R}^{-1} \mathbf{z}\right| \cos(\varphi - \varphi_0)\right) d\varphi \cdot e^{-|\beta|^2 \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}} \\ &= I_0\left(2|\beta| \left|\mathbf{p}^H \mathbf{R}^{-1} \mathbf{z}\right|\right) \cdot e^{-|\beta|^2 \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}} \underset{H_0}{\overset{H_1}{>}} \lambda\end{aligned}$$

where $I_0(x)$ is the modified Bessel function of the 1st kind:

$$I_0(x) \triangleq \frac{1}{2\pi} \int_0^{2\pi} \exp[x(\varphi - \varphi_0)] d\varphi$$

- Since $I_0(x)$ is monotonic in x , the LRT reduces to the **noncoherent WMF**:

$$\left|\mathbf{p}^H \mathbf{R}^{-1} \mathbf{z}\right| \underset{H_0}{\overset{H_1}{>}} \sqrt{\eta}$$

or equivalently

$$\left|\mathbf{p}^H \mathbf{R}^{-1} \mathbf{z}\right|^2 \underset{H_0}{\overset{H_1}{>}} \eta$$

Performance of the non-coherent WMF – SW0

- Performance of the non-coherent WMF when the target amplitude $|\beta|$ is deterministic, i.e. nonfluctuating, and the target phase is uniformly distributed over $[0, 2\pi)$ (Swearing 0).

$$\mathbf{z}|H_0 \in \mathcal{CN}(0, \mathbf{R}), \quad \mathbf{z}|\varphi, H_1 \in \mathcal{CN}(|\beta|e^{j\varphi}\mathbf{p}, \mathbf{R})$$

$$p_{\mathbf{z}|H_1}(\mathbf{z}|H_1) = \frac{1}{2\pi} \int_0^{2\pi} p_{\mathbf{z}|\varphi, H_1}(\mathbf{z}|\varphi, H_1) d\varphi$$

- The **probability of false alarm** is the same as before, since it does not depend on the target signal model:

$$P_{FA} = e^{-\frac{\eta}{\mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}}}$$

Performance of the non-coherent WMF – SW0

- The decision rule can be put in the form:

$$X = |Y|^2 = Y_R^2 + Y_I^2 = \left| \mathbf{p}^H \mathbf{R}^{-1} \mathbf{z} \right|^2 \begin{matrix} > \\ < \end{matrix} \begin{matrix} H_1 \\ H_0 \end{matrix} \eta$$

- We want to calculate:

$$P_D = \int_{\eta}^{+\infty} p_{X|H_1}(x|H_1) dx = \int_{\sqrt{\eta}}^{+\infty} p_{|Y||H_1}(y|H_1) dy$$

PDF of $|Y|$ under the hypothesis H_1

- Let us calculate first the (conditional) PDF of the complex r.v. Y :

$$Y = Y_R + jY_I = \mathbf{p}^H \mathbf{R}^{-1} \mathbf{z}$$

Performance of the non-coherent WMF – SW0

$\Rightarrow Y|\varphi, H_1 = \mathbf{p}^H \mathbf{R}^{-1} \mathbf{z}$ is a complex Gaussian r.v.


$\Rightarrow Y|\varphi, H_1 \in \mathcal{CN}(m_1, \sigma_1^2)$

$$m_1 = E\{Y|\varphi, H_1\} = \mathbf{p}^H \mathbf{R}^{-1} E\{\mathbf{z}|\varphi, H_1\} = \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p} \cdot |\beta| e^{j\varphi}$$

$$\sigma_1^2 = E\{|Y - m_1|^2|\varphi, H_1\} = E\{|\mathbf{p}^H \mathbf{R}^{-1} \mathbf{z} - \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p} \cdot |\beta| e^{j\varphi}|^2|\varphi, H_1\}$$

$$= E\{|\mathbf{p}^H \mathbf{R}^{-1} \mathbf{d}|^2|\varphi, H_1\} = E\{|\mathbf{p}^H \mathbf{R}^{-1} \mathbf{d}|^2\}$$

$$= E\{\mathbf{p}^H \mathbf{R}^{-1} \mathbf{d} \mathbf{d}^H \mathbf{R}^{-1} \mathbf{p}\} = \mathbf{p}^H \mathbf{R}^{-1} E\{\mathbf{d} \mathbf{d}^H\} \mathbf{R}^{-1} \mathbf{p} = \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}$$


$$p_{Y_R, Y_I|H_1}(y_R, y_I|H_1) \equiv p_{Y|H_1}(y|H_1) = \frac{1}{2\pi} \int_0^{2\pi} p_{Y|\varphi, H_1}(y|\varphi, H_1) d\varphi$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\pi \sigma_1^2} \exp\left(-\frac{1}{\sigma_1^2} |y - |\beta| e^{j\varphi} \sigma_1^2|^2\right) d\varphi$$

Performance of the non-coherent WMF – SW0

$$\begin{aligned} p_{Y_R, Y_I | H_1}(y_R, y_I | H_1) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\pi\sigma_1^2} \exp\left(-\frac{|y|^2 + |\beta|^2 \sigma_1^4 - 2|y||\beta|\sigma_1^2 \cos(\varphi - \angle y)}{\sigma_1^2}\right) d\varphi \\ &= \frac{1}{\pi\sigma_1^2} \exp\left(-\frac{|y|^2 + |\beta|^2 \sigma_1^4}{\sigma_1^2}\right) \cdot \frac{1}{2\pi} \int_0^{2\pi} \exp(2|y||\beta|\cos(\varphi - \angle y)) d\varphi \\ &= \frac{1}{\pi\sigma_1^2} \exp\left(-\frac{|y|^2 + |\beta|^2 \sigma_1^4}{\sigma_1^2}\right) I_0(2|y||\beta|) \end{aligned}$$

- Now, to derive the PDF of $|Y|$, we need to consider the following 2-D transformation of r.v.'s (i.e. from Cartesian to polar coordinates):

$$\begin{cases} R = |Y| = \sqrt{Y_R^2 + Y_I^2} \\ \vartheta = \angle Y = \arctan(Y_I / Y_R) \end{cases}$$

Performance of the non-coherent WMF – SW0

- From the well-known theorem of r.v. transformations:

$$p_{|Y|, \angle Y | H_1}(|y|, \angle y | H_1) \equiv p_{R, \vartheta | H_1}(r, \vartheta | H_1) = \frac{p_{Y_R, Y_I | H_1}(y_R, y_I | H_1)}{|\mathbf{J}|} \begin{cases} y_R = r \cos \vartheta \\ y_I = r \sin \vartheta \end{cases}$$

- Where \mathbf{J} is the Jacobian of the transformation: $|\mathbf{J}| = 1/r$

$$p_{R, \vartheta | H_1}(r, \vartheta | H_1) = \frac{r}{\pi \sigma_1^2} \exp\left(-\frac{r^2 + |\beta|^2 \sigma_1^4}{\sigma_1^2}\right) I_0(2r|\beta|), \quad 0 \leq \vartheta < 2\pi, r \geq 0$$

$$\begin{aligned} p_{R | H_1}(r | H_1) &= \int_0^{2\pi} p_{R, \vartheta | H_1}(r, \vartheta | H_1) d\vartheta \\ &= \frac{2r}{\sigma_1^2} \exp\left(-\frac{r^2 + |\beta|^2 \sigma_1^4}{\sigma_1^2}\right) I_0(2r|\beta|), \quad r \geq 0 \end{aligned}$$

Performance of the non-coherent WMF – SW0

- The PDF of the envelope $R=|Y|$ under the hypothesis H_1 is a Rician function:

$$p_{R|H_1}(r|H_1) = \frac{2r}{\sigma_1^2} \exp\left(-\frac{r^2 + |\beta|^2 \sigma_1^4}{\sigma_1^2}\right) I_0(2r|\beta|), \quad r \geq 0,$$

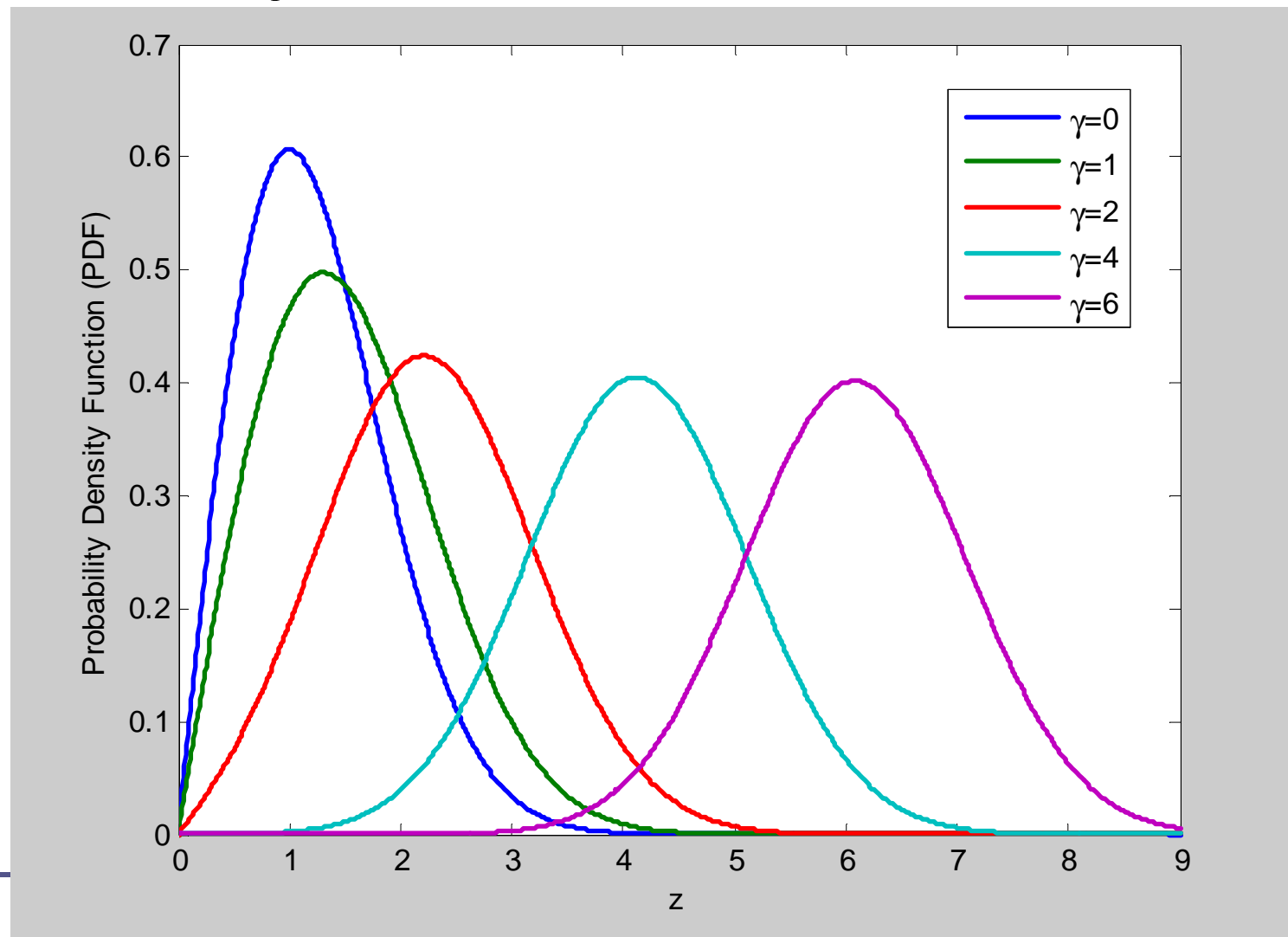
$$SNR_{WMF} = |\beta|^2 \sigma_1^2 = |\beta|^2 \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}$$

- Under the hypothesis H_0 , the PDF is a Rayleigh function; it can be obtained from the Rice PDF by setting $\beta=0$:

$$p_{R|H_0}(r|H_0) = \frac{2r}{\sigma_1^2} \exp\left(-\frac{r^2}{\sigma_1^2}\right), \quad r \geq 0, \quad \sigma_1^2 = \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}$$

Performance of the non-coherent WMF – SW0

$$z \triangleq \frac{\sqrt{2} r}{\sigma_1}, \gamma \triangleq \sqrt{2 \text{SINR}_{\text{WMF}}}, \text{SINR}_{\text{WMF}} = |\beta|^2 \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}$$



Performance of the non-coherent WMF – SW0

$$p_{R|H_1}(r|H_1) = \frac{2r}{\sigma_1^2} \exp\left(-\frac{r^2 + |\beta|^2 \sigma_1^4}{\sigma_1^2}\right) I_0(2r|\beta|), \quad r \geq 0$$

$$P_D = \int_{\frac{\lambda}{\sqrt{\eta}}}^{+\infty} p_{R|H_1}(r|H_1) dr = \int_{\frac{\lambda}{\sqrt{\eta}}}^{+\infty} \frac{2r}{\sigma_1^2} \exp\left(-\frac{r^2 + |\beta|^2 \sigma_1^4}{\sigma_1^2}\right) I_0(2r|\beta|) dr$$

$$= \int_{\lambda}^{+\infty} z \exp\left(-\frac{z^2 + \gamma^2}{2}\right) I_0(z\gamma) dz = Q_M(\gamma, \lambda)$$

where we defined $z \triangleq \frac{\sqrt{2} r}{\sigma_1}$, $\gamma \triangleq \sqrt{2} |\beta| \sigma_1 \Rightarrow z\gamma = 2r|\beta|$, $\lambda = \sqrt{2\eta/\sigma_1^2}$

$Q_M(\gamma, \lambda)$ is the so-called **Marcum Q - function**

Performance of the non-coherent WMF – SW0

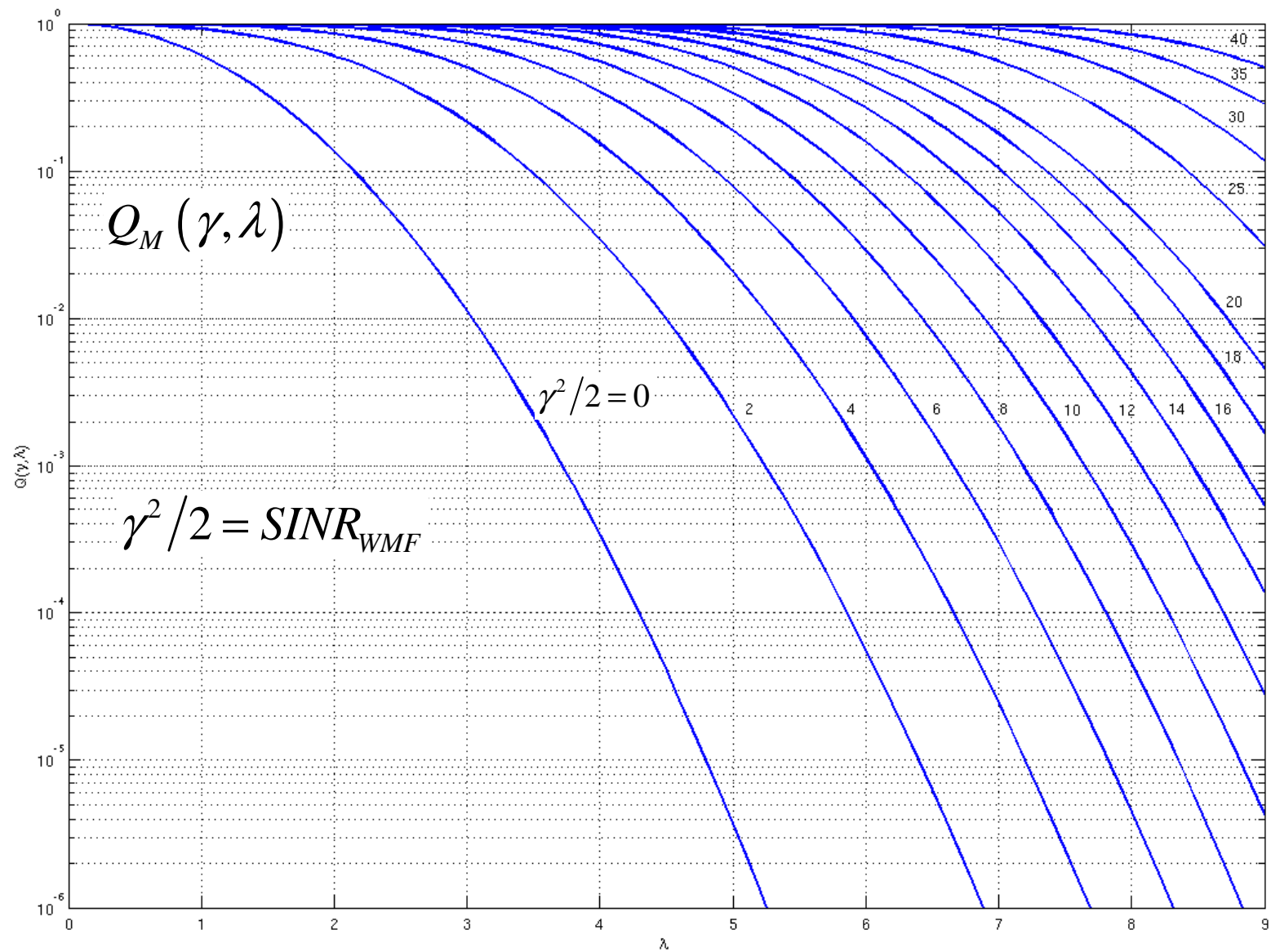
Summarizing:

$$P_{FA} = \exp\left(-\frac{\eta}{\mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}}\right)$$

Note that: $SNR_{WMF} = \frac{SNR_{CWMF}}{2}$

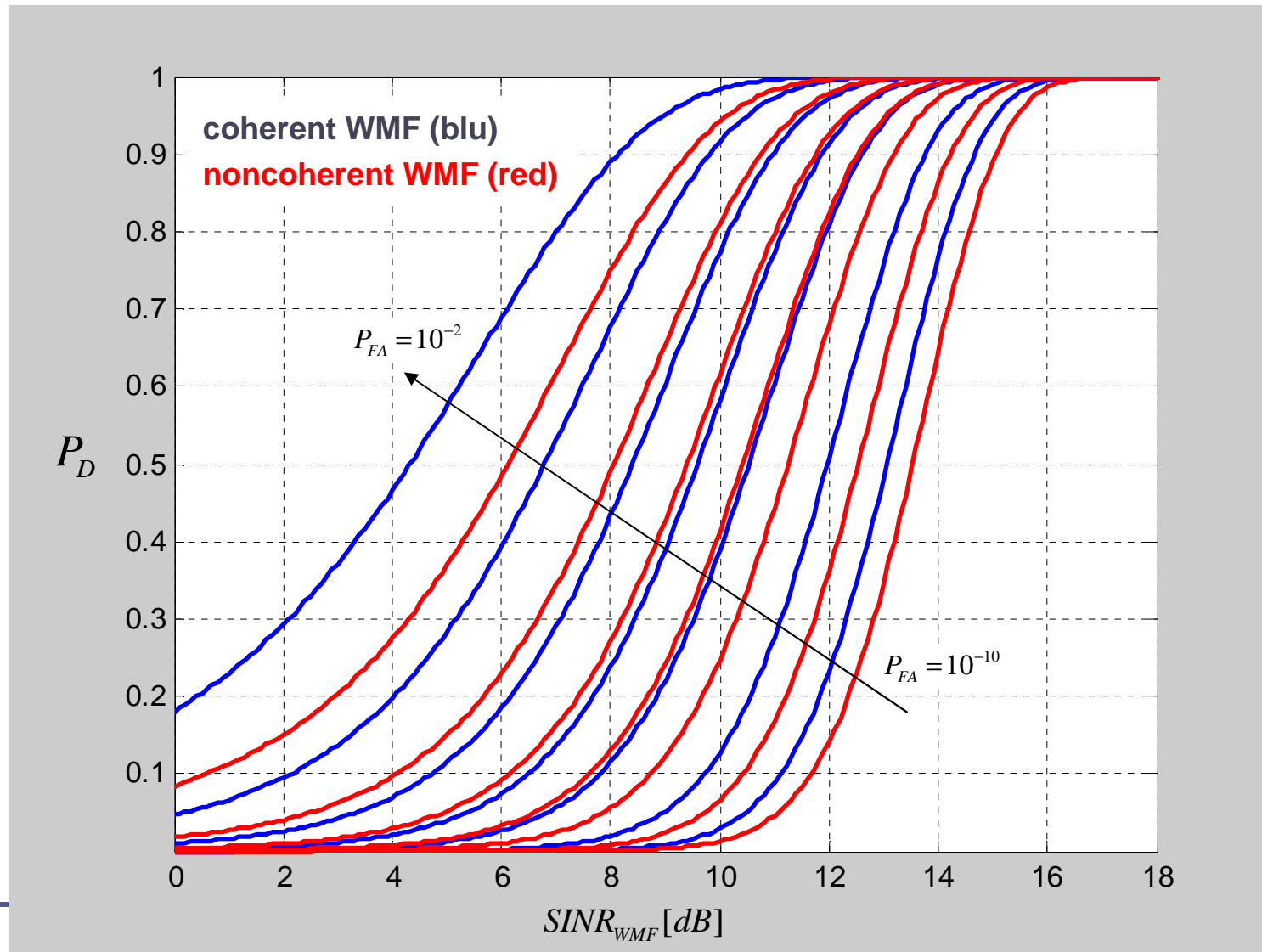
$$P_D = Q_M\left(\sqrt{2|\beta|^2 \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}}, \sqrt{\frac{2\eta}{\mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}}}\right) = Q_M\left(\sqrt{2SNR_{WMF}}, \sqrt{-2\ln(P_{FA})}\right)$$

Performance of the non-coherent WMF – SW0



Performance of the WMF

ROC: the noncoherent WMF (SW-0) vs. coherent WMF.



White Gaussian Noise

- The complex multidimensional PDF of Gaussian disturbance is given by:

$$p_{\mathbf{z}|H_0}(\mathbf{z}|H_0) = p_{\mathbf{d}}(\mathbf{z}) = \frac{1}{(\pi\sigma_d^2)^N} \exp\left(-\frac{\mathbf{z}^H \mathbf{z}}{\sigma_d^2}\right) \quad \mathbf{z}|H_0 \in \mathcal{CN}(\mathbf{0}, \sigma_d^2 \mathbf{I})$$

Nx1 data vector

$$p_{\mathbf{z}|H_1}(\mathbf{z}|H_1) = E_{\mathbf{s}_t} \left\{ p_{\mathbf{z}|H_0}(\mathbf{z} - \mathbf{s}_t|H_0) \right\} = \int \underbrace{\frac{1}{(\pi\sigma_d^2)^N} \exp\left(-\frac{(\mathbf{z} - \mathbf{s}_t)^H (\mathbf{z} - \mathbf{s}_t)}{\sigma_d^2}\right)}_{p_{\mathbf{z}|H_0}(\mathbf{z} - \mathbf{s}_t|H_0)} \cdot p_{\mathbf{s}_t}(\mathbf{s}_t) d\mathbf{s}_t$$

Perfectly Known Target Signal - Case #1

- The optimal NP detection strategy is:

$$l(\mathbf{z}) = \frac{2}{\sigma_d^2} \Re \left\{ \mathbf{s}_t^H \mathbf{z} \right\} \underset{H_0}{>} \underset{H_1}{<} \eta$$

$$P_{FA} = Q \left(\frac{\eta}{\sqrt{SNR_{CWMF}}} \right) \Rightarrow \eta = \sqrt{SNR_{CWMF}} \cdot Q^{-1}(P_{FA})$$

$$P_D = Q \left(\frac{\eta - SNR_{CWMF}}{\sqrt{SNR_{CWMF}}} \right) \Rightarrow P_D = Q \left(Q^{-1}(P_{FA}) - \sqrt{SNR_{CWMF}} \right)$$

where $SNR_{CWMF} = \frac{2}{\sigma_d^2} \mathbf{s}_t^H \mathbf{s}_t$

Swerling 0 Target Signal – Case #3

- If the amplitude is non fluctuating and the phase is uniformly distributed:

$$\begin{aligned}\beta = |\beta|e^{j\varphi} &\Rightarrow |\beta| \text{ is deterministic (nonfluctuating)} \\ &\varphi \text{ is uniformly distributed on } [0, 2\pi) \\ \mathbf{z}|\varphi, H_1 &\in \mathcal{CN}\left(|\beta|e^{j\varphi}\mathbf{p}, \sigma_d^2\mathbf{I}\right)\end{aligned}$$

$$\begin{aligned}\Lambda(\mathbf{z}) &= \frac{1}{2\pi} \int_0^{2\pi} \exp\left(\frac{2|\beta|}{\sigma_d^2} |\mathbf{p}^H \mathbf{z}| \cos(\varphi - \varphi_0)\right) d\varphi \cdot e^{-\frac{|\beta|^2}{\sigma_d^2} N} \\ &= I_0\left(\frac{2|\beta|}{\sigma_d^2} |\mathbf{p}^H \mathbf{z}|\right) \cdot e^{-\frac{|\beta|^2}{\sigma_d^2} N} \begin{matrix} H_1 \\ > \\ < \\ H_0 \end{matrix} \lambda\end{aligned}$$

$$\text{where } \mathbf{p}^H \mathbf{p} = N$$

Swerling 0 Target Signal – Case #3

Since $I_0(x)$ is monotonic in x , the LRT reduces to:

$$\left| \mathbf{p}^H \mathbf{z} \right| \begin{matrix} > \\ < \end{matrix} \begin{matrix} H_1 \\ H_0 \end{matrix} \sqrt{\eta}$$

or equivalently

$$\left| \mathbf{p}^H \mathbf{z} \right|^2 \begin{matrix} > \\ < \end{matrix} \begin{matrix} H_1 \\ H_0 \end{matrix} \eta$$

ROC:
$$P_{FA} = \exp\left(-\frac{\eta\sigma_d^2}{N}\right) \quad P_D = Q_M\left(\sqrt{2SNR_{WMF}}, \sqrt{-2\ln(P_{FA})}\right)$$

where:
$$SNR_{WMF} = \frac{N|\beta|^2}{\sigma_d^2}$$

The **integration** in this case is said **coherent**, because all the pulse samples are first weighed by \mathbf{p} and then coherently summed up. The modulo is taken after summation. **The gain in the SNR for coherent integration is equal to N .**

Swerling 0 vs Swerling 1

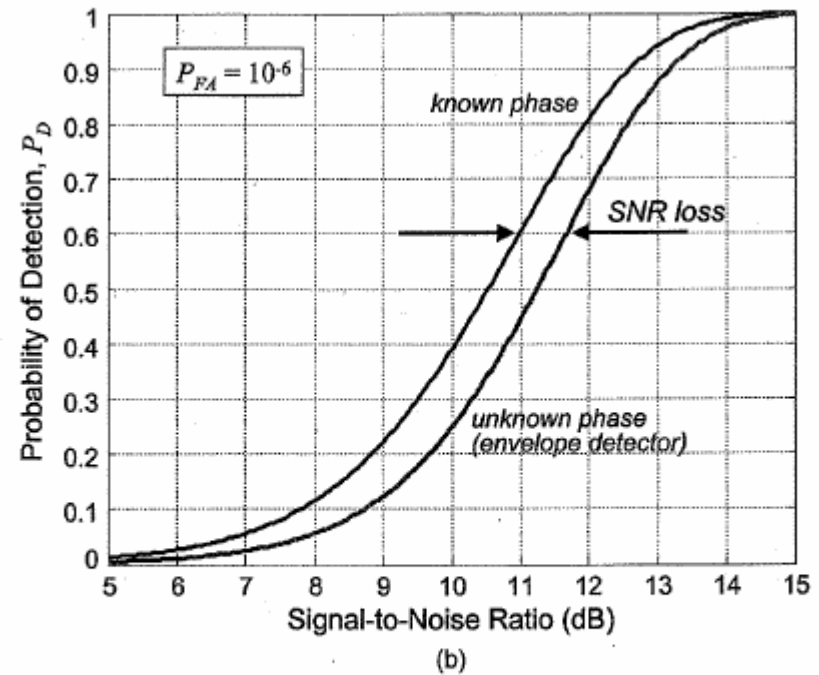
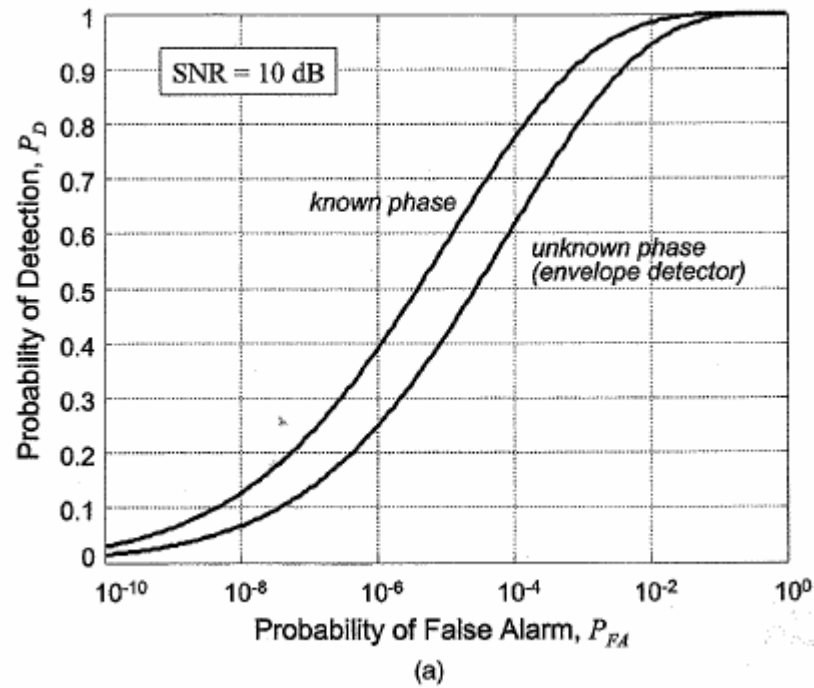


Figure 6.7 Performance difference between coherent and envelope detectors for the Gaussian example. (a) Difference in P_D for $\chi = 10$ dB. (b) Difference in P_D for $P_{FA} = 10^{-6}$.

Incoherent pulse train - Incoherent integration

$$\begin{cases} z(n) = d(n) & H_0 : \text{Target absent} \\ z(n) = Ae^{j\phi(n)} + d(n) & H_1 : \text{Target present} \end{cases}$$

$$\begin{aligned} \Lambda(\mathbf{z}) &= \frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = \frac{\int_{-\infty}^{+\infty} p_{\mathbf{z},\phi|H_1}(\mathbf{z},\phi|H_1)d\phi}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} \\ &= \frac{\prod_{n=0}^{N-1} \int_{-\infty}^{+\infty} p(z(n)|\phi, H_1) p_{\phi|H_1}(\phi|H_1) d\phi}{\prod_{n=0}^{N-1} p(z(n)|H_0)} = \prod_{n=0}^{N-1} \Lambda(z(n)) \end{aligned}$$

$$\Lambda(z(n)) = I_0\left(\frac{2A|z(n)|}{\sigma_d^2}\right) \cdot e^{-\frac{A^2}{\sigma_d^2}}$$

Incoherent pulse train

Applying the logarithm: $\ln \Lambda(\mathbf{z}) = \sum_{n=0}^{N-1} \ln \Lambda(z(n))$

$$l(\mathbf{z}) = \sum_{n=0}^{N-1} \ln I_0 \left(\frac{2A|z(n)|}{\sigma_d^2} \right) \underset{H_0}{>} \underset{H_1}{\eta}$$

The calculation of I_0 is not easy,
so we have to resort to some
approximation

$$\ln I_0(x) \cong \frac{x^2}{4} \quad \text{for } x \ll 1$$

$$\ln I_0(x) \cong x \quad \text{for } x \gg 1$$

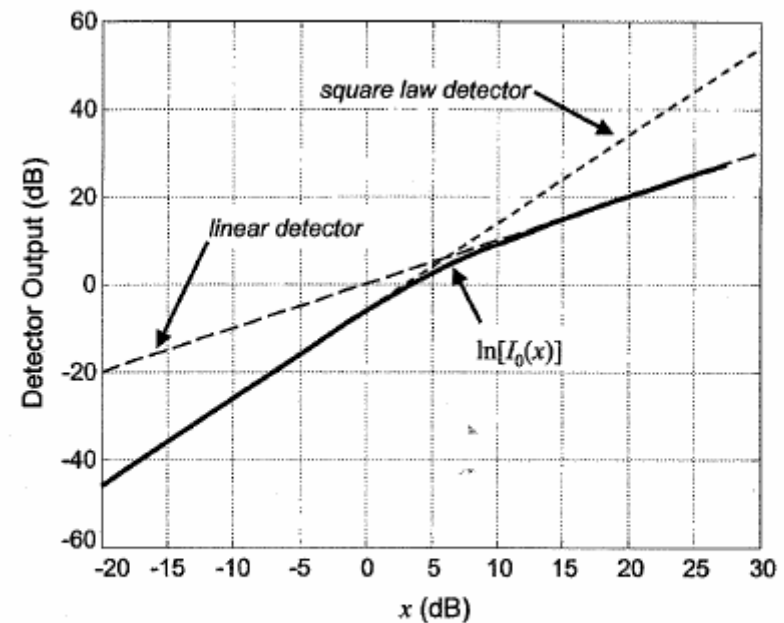


Figure 6.8 Approximation of the $\ln[I_0]$ detector characteristic by the square law detector when its argument is small, and the linear detector when its argument is large.

Incoherent pulse train

Then for $x < 1$ (small SNR) the detector becomes:

$$\sum_{n=0}^{N-1} |z(n)|^2 \underset{H_0}{\overset{H_1}{>}} \eta$$

For $x \gg 1$ (large SNR) the detector becomes:

$$\sum_{n=0}^{N-1} |z(n)| \underset{H_0}{\overset{H_1}{>}} \eta$$

Both detectors perform an incoherent integration, since they first calculate the modulus of each sample, then sum up the results.

The linear approximation fits the $\ln I_0$ very well for $x > 10\text{dB}$, the square law detector is an excellent fit for $x < 5\text{dB}$.

Incoherent pulse train

The integration gain for coherent processing is N

For the incoherent integration, the calculation is not so easy because in general it depends on N , PD and PFA.

For very large values of N , the gain of the quadratic detector can be approximated (in dB) with $10\log_{10} \sqrt{N} - 5.5$. As a rule of thumbs we use \sqrt{N} .

For the linear detector

Albersheim's equation is used.

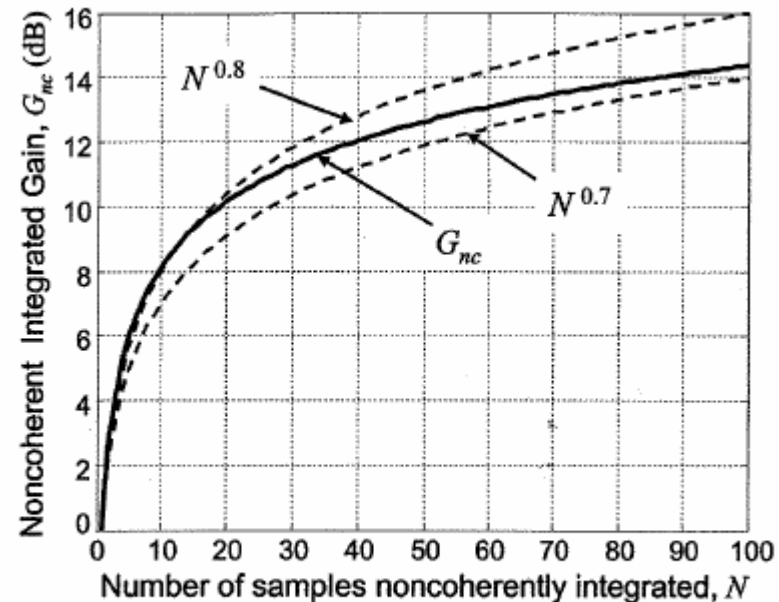


Figure 6.11 Noncoherent integration gain G_{nc} for a nonfluctuating target, as estimated using Albersheim's equation.

Binary integration

- Any coherent or incoherent detection is given by comparing the detection statistic with a threshold, so the output of the detector is binary in the sense that it takes one of only two possible outcomes.
- If the entire detection process is repeated N times for a given range, Doppler, or angle cell, N binary detections will be available.
- Each decision of “target present” will have some probability P_D of being correct, and prob. P_{FA} of being incorrect. To improve the reliability of the detection decision, the decision rule can require that a target is detected on some number M out of N decisions before it is finally accepted as a valid target detection.
- This process is called binary integration with a “ M of N ” rule.
- With the binary integration both overall P_{FA} and P_D are given by

$$P(M, N) = \sum_{k=M}^N \binom{N}{k} p^k (1-p)^{N-k}$$
