Optimum Coherent Detection in Gaussian Disturbance

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“Coherent Radar Detection in Clutter: Clutter Modelling and Analysis + Adaptive Array Radar Signal Processing”
The binary hypothesis testing problem
The Neyman-Pearson (NP) criterion
The Likelihood Ratio Test (LRT)
The Gaussian detection problem
Target signal models (different degrees of a priori knowledge)
Target model #1 - Perfectly known target signal: the coherent whitening matched filter (CWMF)
Performance of the CWMF: $P_{FA}$, $P_D$, and ROC
The function of a surveillance radar is to ascertain whether targets are present in the data. Given a space–time snapshot $z$ (subscript $l$ has been omitted for ease of notation), the signal processor must make a decision as to which of the two hypotheses is true:

$$\begin{align*}
    z &= d & H_0 & : \text{Target absent} \\
    z &= s_t + d & H_1 & : \text{Target present}
\end{align*}$$

The test is implemented on-line for each range cell under test (CUT).

The detection is a binary hypothesis problem whereby we decide either hypothesis $H_0$ is true and only disturbance is present (decision $D_0$) or hypothesis $H_1$ is true and a target signal is present with disturbance (decision $D_1$).

The associated probabilities are:

$$
\begin{align*}
P_D &= \Pr\{D_1|H_1\}, && P_{Miss} = \Pr\{D_0|H_1\} = 1 - P_D, \\
P_{FA} &= \Pr\{D_1|H_0\}, && P_{No\ Target} = \Pr\{D_0|H_0\} = 1 - P_{FA}.
\end{align*}
$$
According to the **Neyman-Pearson (NP) criterion** (maximize $P_D$ while keeping constant $P_{FA}$), the optimal decision strategy is a **likelihood ratio test (LRT)**:

$$\Lambda(z) = \frac{p_{z|H_1}(z|H_1)}{p_{z|H_0}(z|H_0)} \overset{H_1}{>}_0 e^\eta \quad \Rightarrow \quad \ln \Lambda(z) = \ln \left( \frac{p_{z|H_1}(z|H_1)}{p_{z|H_0}(z|H_0)} \right) \overset{H_1}{>}_0 \eta$$

$p_{z|H_i}(z|H_i)$ is the probability density function (PDF) of the random vector $z$ under the hypothesis $H_i$, $i=0,1$.

$\eta$ is the detection threshold (set according to the desired $P_{FA}$).

Disturbance $d$ encompasses any interference or noise component of the data: 

**clutter, jamming, and noise:** $d = c + j + n$

The three components are usually assumed to be mutually **uncorrelated**.
A radar receiver performs linear averaging at the antenna (remember that the antenna pattern integrates everything in its pattern), IF bandpass filters, baseband anti-aliasing filters, pulse compression filters, etc. The Central-Limit Theorem (CLT) therefore applies and the output tends to be Gaussian (even when the input is not Gaussian).

The three disturbance components are assumed zero-mean complex circular Gaussian, mutually uncorrelated, therefore the disturbance covariance matrix is:

$$R = R_c + R_j + R_n$$

(we omit subscript $d$ for ease of notation).

**Optimal processing** (2-D space-time or 1-D spatial (angle) or 1-D temporal (Doppler) processing) refers to the case of a priori known disturbance covariance matrix $R$. The resulting performance represents a benchmark for any realistic (adaptive) approach.

**Adaptive processing** refers to the case where the disturbance matrix (or in general the disturbance PDF parameters) is unknown and must be estimated on-the-fly from the observed data in a CPI.
The complex multidimensional PDF of Gaussian disturbance is given by:

\[
p_{z|H_0}(z|H_0) = p_d(z) = \frac{1}{\pi^{NM} |R|} \exp \left( -z^H R^{-1} z \right)
\]

\[z|H_0 \in \mathcal{CN}(0, R)\]

- \text{The complex multidimensional PDF of target + disturbance:} \quad p_{z|H_1}(z|H_1) = ?

- It depends on the target signal model: \quad z = s_t + d

- If the target vector is deterministic:

\[
p_{z|H_1}(z|H_1) = p_{z|H_0}(z - s_t|H_0)
\]
If the target vector is random with known PDF:

\[
p_{z|H_1}(z|H_1) = E_s \left\{ p_{z|H_0}(z-s_t|H_0) \right\} = \int \frac{1}{\pi^{NM}|R|} \exp \left( -(z-s_t)^H R^{-1} (z-s_t) \right) \cdot p_{s_t}(s_t) ds_t
\]

Under some assumptions, we have found for the target signal:

\[
s_t = \beta v(v_d)
\]

\[N \times 1\]

\[\beta \equiv \alpha_r \text{ is the target complex amplitude}\]

\[
v(v_d) \quad \text{Steering Vector}
\]

\[N \times 1\]

\[v_d = \frac{2v_l T_r}{\lambda} \quad \text{Doppler frequency} \Leftrightarrow \text{target radial velocity}\]
The detection algorithm is optimized for a specific Doppler.

The steering vector should be known (calibrated array).

Since the target velocity os unknown a-priori, \( v \) is a known function of unknown parameters, so the radar receiver should implement multiple detectors that form a filter bank to cover all potential target Doppler frequencies.
Different models of $s_t$ have been investigated to take into account different degrees of \textit{a priori} knowledge on the target signal:

(1) $s_t$ perfectly known $\rightarrow$ \textbf{coherent whitening matched filter} (CWMF)

(2) $s_t = \beta v$ with $\beta \in \mathcal{CN}(0, \sigma_s^2)$, i.e., Swerling I model, and $v$ perfectly known;

(3) $s_t = \beta v$ with $|\beta|$ deterministic and $\angle \beta$ random, uniformly distributed in $[0, 2\pi)$, i.e. Swerling 0 (or Swerling V) model, and $v$ perfectly known

(4) $s_t = \beta v$ with $\beta$ unknown deterministic and $v$ perfectly known;
The optimal NP decision strategy is a LRT (or log-LRT):

\[
l(z) = \ln \Lambda(z) = \ln \frac{p_{z|H_1}(z|H_1)}{p_{z|H_0}(z|H_0)} = z^H R^{-1}z - (z - s_t)^H R^{-1}(z - s_t)
\]

\[
= s_t^H R^{-1}z + z^H R^{-1}s_t - s_t^H R^{-1}s_t = 2\Re \{ s_t^H R^{-1}z \} - s_t^H R^{-1}s_t > \eta
\]

It is the so-called coherent whitening matched filter (CWMF) detector:

\[
s_t^H R^{-1}z = s_t^H R^{-1/2} R^{-1/2}z = s_t^H \left( R^{-1/2} \right)^H R^{-1/2}z = \left( R^{-1/2} s_t \right)^H R^{-1/2}z = \overline{s}_t^H \overline{z}
\]

\[
\overline{s}_t \triangleq R^{-1/2} s_t, \quad \overline{z} \triangleq R^{-1/2} z
\]

whitening transformation

\[
E \left\{ \overline{z} \overline{z}^H \right\} = E \left\{ R^{-1/2} z z^H R^{-1/2} \right\} = R^{-1/2} E \left\{ z z^H \right\} R^{-1/2} = R^{-1/2} RR^{-1/2} = I
\]
The **WMF** is a linear filter: 

$$ s_t^H R^{-1} z = w^H z = \sum_{k=1}^{MN} W_k^* z_k = \sum_{n=1}^{N} \sum_{m=1}^{M} W_{n,m}^* z_{n,m} $$

- **w** is the optimal weighting vector: 
  
  $$ w^H = s_t^H R^{-1} \Rightarrow w = R^{-H} s_t = R^{-1} s_t $$

- $w = R^{-1} s_t$

- $s_t = \beta v(v_d, v_s)$

- $\delta \triangleq s_t^H R^{-1} s_t$

- $l(z) = 2 \Re \{ y \} - \delta > \eta$
To compare the performance of different non-adaptive algorithms with each other, as well as with adaptive approaches, it is necessary to use some standard benchmarks.

Important radar performance metrics are the probability of detecting a target \( P_D \), the probability of declaring a false alarm \( P_{FA} \), and the accuracy with which target speed and/or bearing may be measured.

Useful intermediate quantities for the probability of detection are the signal-to-interference-plus-noise power ratio (SINR) and SINR loss (defined later on).

Equivalently, sometimes we find the signal-to-disturbance power ratio (SDR).

Minimum Detectable Velocity. The width of the SINR loss notch near mainlobe clutter determines the lowest velocity detectable by the radar.

Finally, the response of filter \( w \) itself is important: it should have a distinct mainlobe that is as narrow as possible as well as low sidelobes.
Performance of the coherent WMF

\[ l(z) = 2\Re\{s_t^H R^{-1} z\} - s_t^H R^{-1} s_t > \eta \]

\[ z \mid H_0 \in \mathcal{CN}(0, R), \quad z \mid H_1 \in \mathcal{CN}(s_t, R) \implies l \text{ is a real Gaussian r.v.} \]

\[ m_0 = E\{l \mid H_0\} = 2\Re\{s_t^H R^{-1} E\{z \mid H_0\}\} - s_t^H R^{-1} s_t = -s_t^H R^{-1} s_t \]

\[ m_1 = E\{l \mid H_1\} = 2\Re\{s_t^H R^{-1} E\{z \mid H_1\}\} - s_t^H R^{-1} s_t = s_t^H R^{-1} s_t \]

- The fact that \( d \) is a complex circular Gaussian random vector implies that [Ch.13, Kay98]:

\[ E\{dd^H\} = R = R^H \quad \text{and} \quad E\{dd^T\} = 0 \]

\[
\sigma_0^2 = \text{var}\{l \mid H_0\} = E\left\{\left|l - E\{l \mid H_0\}\right|^2 \mid H_0\right\} \\
= E\left\{\left|2\Re\{s_i^H R^{-1} z\}\right|^2 \mid H_0\right\} = E\left\{\left|s_i^H R^{-1} z + z^H R^{-1} s_i\right|^2 \mid H_0\right\} \\
= E\left\{\left(s_i^H R^{-1} d + d^H R^{-1} s_i\right)\left(s_i^H R^{-1} d + d^H R^{-1} s_i\right)^H\right\} \\
= E\left\{s_i^H R^{-1} s_i^H R^{-1} d + s_i^H R^{-1} d d^H R^{-1} s_i + d^H R^{-1} s_i s_i^H R^{-1} d + d^H R^{-1} s_i d^H R^{-1} s_i\right\} \\
= E\left\{s_i^H R^{-1} d d^T R^{-1} s_i^* + 2s_i^H R^{-1} d d^H R^{-1} s_i + s_i^T R^{-1} d^* d^H R^{-1} s_i\right\} \\
= E\left\{2s_i^H R^{-1} d d^T R^{-1} s_i^* + 2s_i^H R^{-1} d d^H R^{-1} s_i\right\} \\
= 2s_i^H R^{-1} E\{d d^T\} R^{-1} s_i^* + 2s_i^H R^{-1} E\{d d^H\} R^{-1} s_i \\
= 2s_i^H R^{-1} R R^{-1} s_i = 2s_i^H R^{-1} s_i \\
\sigma_i^2 = \text{var}\{l \mid H_i\} = E\left\{\left|l - E\{l \mid H_i\}\right|^2 \mid H_i\right\} = 2s_i^H R^{-1} s_i,
\]

we used the fact that \(s_i^H R^{-1} d = (s_i^H R^{-1} d)^T = d^T R^{-T} s_i^* = d^T R^{-1} s_i^*\) and \(E\{d d^T\} = 0\)
The probability of detection \((P_D)\) and probability of false alarm \((P_{FA})\)
can be calculated using the statistics we have just derived:

\[
P_{FA} = \Pr\{l > \eta \mid H_0\} = \int_{\eta}^{+\infty} p_{l\mid H_0}(l \mid H_0) dl = \int_{\eta}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma_0^2} \exp\left(-\frac{(l - m_0)^2}{2\sigma_0^2}\right) dl
\]

\[
x = \frac{l - m_0}{\sigma_0}
\]

\[
= \int_{\frac{\eta - m_0}{\sigma_0}}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx = Q\left(\frac{\eta - m_0}{\sigma_0}\right) = Q\left(\frac{\eta + s_t^H R^{-1} s_t}{\sqrt{2s_t^H R^{-1} s_t}}\right) = Q\left(\frac{\eta}{d} + \frac{d}{2}\right)
\]

\[
P_D = \Pr\{l > \eta \mid H_1\} = \int_{\eta}^{+\infty} p_{l\mid H_1}(l \mid H_1) dl = Q\left(\frac{\eta - m_1}{\sigma_1}\right) = Q\left(\frac{\eta - s_t^H R^{-1} s_t}{\sqrt{2s_t^H R^{-1} s_t}}\right) = Q\left(\frac{\eta}{d} - \frac{d}{2}\right)
\]

where \(d \triangleq \frac{|m_1 - m_0|}{\sigma_0} = \frac{2s_t^H R^{-1} s_t}{\sqrt{2s_t^H R^{-1} s_t}} = \sqrt{2s_t^H R^{-1} s_t}\)

- What is the meaning of \(d\)?

---

**Performance of the coherent WMF**
The filter output is:  
\[ y = w^H z = w^H (s_t + d) = w^H s_t + w^H d \quad (H_t) \]

The **Signal-to-Interference plus Noise ratio (SINR)** at the output of \( w \) is:

\[ \text{SINR} = \frac{E\left\{ |w^H s_t|^2 \right\}}{E\left\{ |w^H d|^2 \right\}} = \frac{E\left\{ w^H s_t s_t^H w \right\}}{E\left\{ w^H d d^H w \right\}} = \frac{w^H E\{s_t s_t^H\} w}{w^H E\{d d^H\} w} = \frac{w^H s_t s_t^H w}{w^H R w} = \frac{|w^H s_t|^2}{w^H R w} \]

The **SINR** at the output of the optimal WMF:  
\[ w = R^{-1} s_t \]

\[ \text{SINR}_{\text{WMF}} = s_t^H R^{-1} s_t \]

For perfectly known target signal, we take the real part of the WMF output, therefore the disturbance power is halved:

\[ \Re\{s_t^H R^{-1} z\} = \Re\{s_t^H R^{-1} s_t\} + \Re\{s_t^H R^{-1} d\} = s_t^H R^{-1} s_t + \Re\{s_t^H R^{-1} d\} \]
\[ \Re\{s_t^H R^{-1}z\} = s_t^H R^{-1}s_t + \Re\{s_t^H R^{-1}d\} \]

Therefore, the SINR at the output of the coherent WMF (CWMF) is:

\[
\text{SINR}_{\text{CWMF}} = \frac{\left( s_t^H R^{-1}s_t \right)^2}{E\left\{ \left( \Re\{s_t^H R^{-1}d\} \right)^2 \right\}} = \frac{\left( s_t^H R^{-1}s_t \right)^2}{E\left\{ \frac{s_t^H R^{-1}d + d^H R^{-1}s_t}{2} \right\}^2}
\]

\[
= \frac{4\left( s_t^H R^{-1}s_t \right)^2}{2E\left\{ s_t^H R^{-1}d \right\}^2} = \frac{4\left( s_t^H R^{-1}s_t \right)^2}{2s_t^H R^{-1}s_t} = 2s_t^H R^{-1}s_t = d^2
\]

therefore \( d = \sqrt{2s_t^H R^{-1}s_t} = \sqrt{\text{SINR}_{\text{CWMF}}} \)
Performance of the coherent WMF

- $P_D$ and $P_{FA}$ completely specify the Neyman-Pearson test performance.
- Insight is provided by plotting these probabilities against one another, this is called the **Receiver Operating Characteristic (ROC) curve**:

\[
P_{FA} = Q\left(\frac{\eta}{d} + \frac{d}{2}\right) \Rightarrow \eta = d \cdot \left(Q^{-1}(P_{FA}) - \frac{d}{2}\right)
\]

\[
P_D = Q\left(\frac{\eta}{d} - \frac{d}{2}\right) \Rightarrow P_D = Q\left(Q^{-1}(P_{FA}) - d\right)
\]

where $d^2 = 2s_i^H R^{-1} s_i$ is the SINR at the output of the coherent WMF.

- Note that $P_D$ is a monotonic increasing function of the SINR at the output of the optimal filter, therefore to maximize the SINR is equivalent to maximize $P_D$.

- In Matlab: see qfunc.m and qfuncinv.m
Performance of the coherent WMF

Receiver Operating Characteristic (ROC) curves:

The ROC is a function of only the SINR and the threshold.

Here, the threshold is varied to cover the all range of interest and the ROC curve is plotted for four different SINR values.

\[ \zeta = \text{SINR}_{\text{CWMF}} = d^2 \]

CWMF_ROC.m
Swerling I Target Signal - Case #2

\[ s_i = \beta v(v_d), \quad \beta \in CN\left(0, \sigma_s^2\right), \quad v \text{ perfectly known} \]

- **v perfectly known** means: array perfectly calibrated, I & Q channels perfectly matched (i.e. balanced), and we are testing for the presence of a target in a given DOA and Doppler frequency bin.

\[ \beta = |\beta| e^{j\phi} \in CN\left(0, \sigma_s^2\right) \Rightarrow |\beta| \text{ is Rayleigh distributed, } E\{|\beta|^2\} = \sigma_s^2 \]

\[ \phi \text{ is uniformly distributed on } [0,2\pi) \]

- Based on these assumptions:

\[ z|H_0 \in CN\left(0, R\right), \quad z|H_1 \in CN\left(0, \sigma_s^2 vv^H + R\right) \]

\[ E\{zz^H | H_1\} = E\{(\beta v + d)(\beta v + d)^H\} = E\{|\beta|^2\} vv^H + R = \sigma_s^2 vv^H + R \]
The log-likelihood ratio (log-LR or LLR) is given by:

$$p_{z|H_0}(z|H_0) = \frac{1}{\pi^{NM}|R|} \exp(-z^H R^{-1} z)$$

$$p_{z|H_1}(z|H_1) = \frac{1}{\pi^{NM}|\sigma_s^2 vv^H + R|} \exp(-z^H (\sigma_s^2 vv^H + R)^{-1} z)$$

The log-likelihood ratio (log-LR or LLR) is given by:

$$\ln \frac{p_{z|H_1}(z|H_1)}{p_{z|H_0}(z|H_0)} = \ln |R| - \ln |\sigma_s^2 vv^H + R| + z^H R z - z^H (\sigma_s^2 vv^H + R)^{-1} z$$

By making use of Woodbury’s identity:

$$(\sigma_s^2 vv^H + R)^{-1} = R^{-1} - \frac{\sigma_s^2 R^{-1} vv^H R^{-1}}{1 + \sigma_s^2 v^H R^{-1} v}$$
By incorporating the non-essential terms in the threshold, we derive that in this case the optimal NP decision strategy is:

\[
\ln \Lambda(z) = \ln |R| - \ln |\sigma_s^2 v v^H + R| + \sigma_s^2 \frac{z^H R^{-1} v v^H R^{-1} z}{1 + \sigma_s^2 v^H R^{-1} v} > \eta
\]

Again, the optimal decision strategy requires calculation of the WMF output, but instead of taking the real part of the output we calculate the modulo of the output: this is due to the fact that the target phase in this case is unknown.

This is sometimes termed the noncoherent WMF.
Swerling I Target Signal - Case #2

**Coherent vs. noncoherent WMF:**

\[
\Re\{s_t^H R^{-1}z\} = \Re\{\beta^* v^H R^{-1}z\} = \Re\{\beta^* v^H R^{-1}(\beta v + d)\} \\
= \Re\{|\beta|^2 v^H R^{-1}v\} + \Re\{\beta^* v^H R^{-1}d\} \\
= |\beta|^2 v^H R^{-1}v + \Re\{\beta^* v^H R^{-1}d\}
\]

We cannot take the real part if we before do not compensate for target phase, in fact if we would do that:

\[
\Re\{v^H R^{-1}z\} = \Re\{\beta v^H R^{-1}v + v^H R^{-1}d\} = v^H R^{-1}v |\beta| \cos(\angle \beta) + \Re\{v^H R^{-1}d\}
\]

Hence, if we do not know the target phase we have to calculate the modulo (squared) of the WMF output:

\[
|v^H R^{-1}z|^2 = |\beta v^H R^{-1}v + v^H R^{-1}d|^2 = |\beta|^2 (v^H R^{-1}v)^2 + \text{disturbance}
\]
If the amplitude is non fluctuating and the phase is uniformly distributed:

$$\beta = |\beta| e^{i\varphi} \quad \Rightarrow \quad |\beta| \text{ is deterministic (nonfluctuating)}$$

$$\varphi \quad \text{is uniformly distributed on } [0,2\pi)$$

$$z|\varphi, H_1 \in CN\left(|\beta|e^{i\varphi} \mathbf{v}, \mathbf{R}\right)$$

$$\Lambda(z) = \frac{p_{z|H_1}(z|H_1)}{p_{z|H_0}(z|H_0)} = \frac{\int_{-\infty}^{+\infty} p_{z,\varphi|H_1}(z,\varphi|H_1)d\varphi}{p_{z|H_0}(z|H_0)} = \frac{\int_{-\infty}^{+\infty} p_{z|\varphi,H_1}(z|\varphi,H_1)p_{\varphi|H_1}(\varphi|H_1)d\varphi}{p_{z|H_0}(z|H_0)}$$

$$= \frac{\int_{-\infty}^{+\infty} p_{z,\varphi|H_1}(z|\varphi,H_1)p_{\varphi}(\varphi)d\varphi}{p_{z|H_0}(z|H_0)} = \frac{1}{2\pi} \int_{0}^{2\pi} p_{z|\varphi,H_1}(z|\varphi,H_1)d\varphi$$
\[ \Lambda(z) = \frac{1}{2\pi} \frac{\int_0^{2\pi} p_{z|\varphi,H_1}(z|\varphi,H_1) d\varphi}{p_{z|H_0}(z|H_0)} \]

\[
\frac{1}{2\pi} \int_0^{2\pi} \exp\left( -(z - |\beta| e^{j\phi} v)^H R^{-1} (z - |\beta| e^{j\phi} v) \right) d\phi
\]

\[
= \frac{\int_0^{2\pi} \exp\left( |\beta| e^{-j\phi} v^H R^{-1} z + |\beta| e^{j\phi} z^H R^{-1} v - |\beta|^2 v^H R^{-1} v \right) d\phi}{\exp(-z^H R^{-1} z)}
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \exp\left( 2|\beta| v^H R^{-1} z \cos(\varphi - \varphi_0) - |\beta|^2 v^H R^{-1} v \right) d\varphi
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \exp\left( 2|\beta| v^H R^{-1} z \cos(\varphi - \varphi_0) \right) d\varphi \cdot e^{-|\beta|^2 v^H R^{-1} v}
\]

where \( \varphi_0 \triangleq \angle v^H R^{-1} z \)
Swerling 0 Target Signal – Case #3

\[
\Lambda(z) = \frac{1}{2\pi} \int_0^{2\pi} \exp\left(2|\beta||v^H R^{-1}z|\cos(\phi - \phi_0)\right) d\phi \cdot e^{-|\beta|^2 v^H R^{-1}v} \\
= I_0\left(2|\beta||v^H R^{-1}z|\right) \cdot e^{-|\beta|^2 v^H R^{-1}v} > \begin{cases} H_1 \\ < H_0 \end{cases} \left. e^\eta \right.
\]

where \( I_0(x) \) is the modified Bessel function of the 1st kind:

\[
I_0(x) \triangleq \frac{1}{2\pi} \int_0^{2\pi} \exp[x(\phi - \phi_0)] d\phi
\]

- Since \( I_0(x) \) is monotonic in \( x \), the LRT reduces to the **noncoherent WMF**:

\[
\left| v^H R^{-1}z \right|^{2} > \begin{cases} H_1 \\ < H_0 \end{cases} \eta
\]
s_t = \beta v(v_d), \quad \beta \text{ deterministic unknown, } \quad v \text{ perfectly known}

\mathbf{z}|_{H_0} \in \mathbb{C}N(0, \mathbf{R}), \quad \mathbf{z}|_{H_1} \in \mathbb{C}N(\beta v, \mathbf{R})

\Lambda(\mathbf{z}; \beta) = \frac{p_{z|H_1}(\mathbf{z}|H_1; \beta)}{p_{z|H_0}(\mathbf{z}|H_0)} \Rightarrow e^\eta

\text{The LRT depends on the unknown complex target amplitude, therefore it cannot be implemented.}

\text{This is a composite hypothesis testing problem.}

\text{A uniformly most powerful (UMP) test does not exist.}

\text{We resort to the generalized likelihood ratio test (GLRT) \rightarrow the unknown parameters are replaced by their maximum likelihood estimates (MLE).}

\Lambda_{GLRT}(\mathbf{z}) = \max_\beta \Lambda(\mathbf{z}; \beta) = \frac{\max_\beta p_{z|H_1}(\mathbf{z}|H_1; \beta)}{p_{z|H_0}(\mathbf{z}|H_0)} = \frac{p_{z|H_1}(\mathbf{z}|H_1; \hat{\beta}_{ML})}{p_{z|H_0}(\mathbf{z}|H_0)} \Rightarrow e^\eta
The GLRT is a suboptimal detector. However, it usually produces good
detection performance. For large data records it can be shown to be
UMP within the class of invariant detectors [Kay98].

\[
\ln \Lambda(z; \beta) = \ln \frac{p_{z|H_1}(z|H_1; \beta)}{p_{z|H_0}(z|H_0)} = z^H R^{-1}z - (z - \beta v)^H R^{-1}(z - \beta v)
\]

\[
\hat{\beta}_{ML} = \arg \max_{\beta} \left[ z^H R^{-1}z - (z - \beta v)^H R^{-1}(z - \beta v) \right]
= \arg \min_{\beta} \left[ (z - \beta v)^H R^{-1}(z - \beta v) \right]
\]

\[
(z - \beta v)^H R^{-1}(z - \beta v) = z^H R^{-1}z + |\beta|^2 v^H R^{-1}v - 2\Re \{\beta^* z^H R^{-1}v\}
\]

\[
= z^H R^{-1}z + v^H R^{-1}v \cdot \beta - \frac{v^H R^{-1}z}{v^H R^{-1}v} - \frac{|v^H R^{-1}z|^2}{v^H R^{-1}v}
\]

The minimum is clearly attained when
the positive factor containing \(\beta\) is made to vanish.
The MLE of \( \beta \) is unbiased and efficient: the MSE on the estimates of amplitude and phase coincide with their Cramér-Rao lower bounds (CLRBs).
\[(z - \hat{\beta}_{ML} v)^H R^{-1} (z - \hat{\beta}_{ML} v) = z^H R^{-1} z - \frac{|v^H R^{-1} z|^2}{v^H R^{-1} v}\]

As a consequence we find the GLRT to be:

\[
\ln \Lambda_{GLRT}(z) = \ln \frac{p_{z|H_1}(z| H_1; \hat{\beta}_{ML})}{p_{z|H_0}(z| H_0)} = \frac{|v^H R^{-1} z|^2}{v^H R^{-1} v} > \eta_{H_1} \ \ \ \ < \eta_{H_0}
\]

Incorporating the denominator in the threshold, again we find that the decision strategy is to compare the modulo-squared WMF output to a threshold:

\[
|v^H R^{-1} z|^2 > \eta_{H_1} \ \ \ \ < \eta_{H_0}
\]
Performance of the non-coherent WMF: \textbf{Signal-to-Interference plus Noise power ratio (SINR)}

\[
\left| w^H z \right|^2 = \left| v^H R^{-1} z \right|^2 > \eta \\
\Rightarrow w = \kappa R^{-1} v \quad \text{optimal weights}
\]

\[
SINR = \frac{E \left\{ \left| w^H s_i \right|^2 \right\}}{E \left\{ \left| w^H d \right|^2 \right\}} = \frac{E \left\{ \left| w^H s_i \right|^2 \right\}}{w^H R w} \Rightarrow SINR_{WMF} = E \left\{ \left| \beta \right|^2 \right\} v^H R^{-1} v
\]

\[
SINR_{WMF} = \begin{cases} 
\sigma_s^2 v^H R^{-1} v, & \beta \in \mathcal{CN} \left(0, \sigma_s^2 \right) \\
\left| \beta \right|^2 v^H R^{-1} v, & \beta \text{ deterministic}
\end{cases}
\]
(a) Performance of the non-coherent WMF when the target complex amplitude is a zero-mean circular complex Gaussian r.v. (Swerling I).

\[ z | H_0 \in \mathcal{CN}(0, \mathbf{R}), \quad z | H_1 \in \mathcal{CN}(0, \sigma_s^2 \mathbf{v} \mathbf{v}^H + \mathbf{R}) \]

\[ \Rightarrow \quad Y \triangleq \mathbf{v}^H \mathbf{R}^{-1} z \text{ is a complex circular Gaussian r.v. } |H_i| \]

\[ \Rightarrow \quad Y | H_i \in \mathcal{CN}(m_i, \sigma_i^2), i = 0,1 \]

\[ m_0 = E\{Y | H_0\} = \mathbf{v}^H \mathbf{R}^{-1} E\{z | H_0\} = \mathbf{v}^H \mathbf{R}^{-1} E\{\mathbf{d}\} = 0 \]

\[ m_1 = E\{Y | H_1\} = \mathbf{v}^H \mathbf{R}^{-1} E\{z | H_1\} = \mathbf{v}^H \mathbf{R}^{-1} \mathbf{v} \cdot E\{\beta\} + \mathbf{v}^H \mathbf{R}^{-1} E\{\mathbf{d}\} = 0 \]

\[ X = |Y|^2 = Y_R^2 + Y_I^2 \quad \Rightarrow \quad X | H_i \in \text{Exp}(\sigma_i^2), i = 0,1 \]

\[ P_{X|H_i}(x | H_i) = \frac{1}{\sigma_i^2} e^{-\frac{x}{\sigma_i^2}} u(x) \]

\[ u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \]
\[ \sigma_0^2 = \text{var}\{Y|H_0\} = E\{|Y|^2|H_0\} = E\left\{|v^H R^{-1}z|^2|H_0\right\} = E\left\{|v^H R^{-1}d|^2\right\} \\
= E\left\{v^H R^{-1}dd^H R^{-1}v\right\} = v^H R^{-1}v \]

\[ \sigma_1^2 = \text{var}\{Y|H_1\} = E\{|Y|^2|H_1\} = E\left\{|v^H R^{-1}z|^2|H_1\right\} \\
= E\left\{|\beta v^H R^{-1}v + v^H R^{-1}d|^2\right\} = E\left\{|\beta v^H R^{-1}v|^2 + |v^H R^{-1}d|^2\right\} \\
= \sigma_s^2 (v^H R^{-1}v)^2 + v^H R^{-1}v = (v^H R^{-1}v)(1 + \sigma_s^2 v^H R^{-1}v) \]

\[ P_{FA} = \int_{\eta}^{+\infty} p_{X|H_0}(x|H_0)dx = \frac{-\eta}{\sigma_0^2} \]

\[ P_D = \int_{\eta}^{+\infty} p_{X|H_1}(x|H_1)dx = \frac{\eta}{\sigma_1^2} \]
Performance of the non-coherent WMF – SW1

\[ P_{FA} = e^{-\frac{\eta}{\sigma_0^2}} \Rightarrow \eta = \sigma_0^2 \ln\left(\frac{1}{P_{FA}}\right) \]

\[ P_D = e^{-\frac{\eta}{\sigma_1^2}} = e^{-\frac{\sigma_0^2 \ln(1/P_{FA})}{\sigma_1^2}} = \left(e^{-\ln(1/P_{FA})}\right) \frac{\sigma_0^2}{\sigma_1^2} = \left(P_{FA}\right) \frac{\sigma_0^2}{\sigma_1^2} \]

where \[ \frac{\sigma_1^2}{\sigma_0^2} = 1 + \sigma_s^2 v^H R^{-1} v = 1 + SINR_{WMF} \]

- Receiver Operating Characteristic (ROC) curves:

\[ P_D = \left(P_{FA}\right)^{\frac{1}{1+SINR_{WMF}}} \]

- \( P_D \) is a monotonic increasing function of \( SINR_{WMF} \).
(b) Performance of the non-coherent WMF when the target amplitude $|\beta|$ is deterministic, i.e. nonfluctuating, and the target phase is uniformly distributed over $[0,2\pi)$ (Swerling 0/Swerling V).

$$z|H_0 \in CN(0, R), \quad z|\varphi, H_1 \in CN(|\beta|e^{j\varphi} v, R)$$

$$p_{z|H_1}(z|H_1) = \frac{1}{2\pi} \int_0^{2\pi} p_{z|\varphi,H_1}(z|\varphi, H_1) d\varphi$$

The **probability of false alarm** is the same as before, since it does not depend on the target signal model:

$$P_{FA} = e^{-\frac{\eta}{v^H R^{-1} v}}$$
The decision rule can be put in the form:

\[ X = |Y|^2 = Y_R^2 + Y_I^2 = \left| \mathbf{v}^H \mathbf{R}^{-1} \mathbf{z} \right|^2 > \eta \]

We want to calculate:

\[ PD = \int_{\eta}^{+\infty} p_{X|H_1}(x|H_1)dx = \int_{\sqrt{\eta}}^{+\infty} p_{|Y|\mathbb{H}_1}(y|H_1)dy \]

PDF of |\( Y \)| under the hypothesis \( H_1 \)

Let us calculate first the (conditional) PDF of the complex r.v. \( Y \):

\[ Y = Y_R + jY_I = \mathbf{v}^H \mathbf{R}^{-1} \mathbf{z} \]
\[ \Rightarrow Y \mid \phi, H_1 = v^H R^{-1} z \] is a complex Gaussian r.v.

\[ \Rightarrow Y \mid \phi, H_1 \in \mathcal{CN}(m_1, \sigma_1^2) \]

\[ m_1 = E\{Y \mid \phi, H_1\} = v^H R^{-1} E\{z \mid \phi, H_1\} = v^H R^{-1} v \cdot |\beta| e^{i\phi} \]

\[ \sigma_1^2 = E\{|Y - m_1|^2 \mid \phi, H_1\} = E\{|v^H R^{-1} z - v^H R^{-1} v \cdot |\beta| e^{i\phi}|^2 \mid \phi, H_1\} \]

\[ = E\{|v^H R^{-1} d|^2 \mid \phi, H_1\} = E\{|v^H R^{-1} d|^2\} \]

\[ = E\{v^H R^{-1} d d^H R^{-1} v\} = v^H R^{-1} E\{d d^H\} R^{-1} v = v^H R^{-1} v \]

\[ p_{Y_R,Y_I|H_1}(y_R, y_I \mid H_1) \equiv p_{Y|H_1}(y \mid H_1) = \frac{1}{2\pi} \int_0^{2\pi} p_{Y|\phi,H_1}(y \mid \phi, H_1) d\phi \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\pi \sigma_1^2} \exp \left( -\frac{1}{\sigma_1^2} |y - |\beta| e^{i\phi} \sigma_1^2|^2 \right) d\phi \]
Performance of the non-coherent WMF – SW0

\[
p_{Y_R,Y_I|H_1}(y_R, y_I | H_1) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\pi \sigma_1^2} \exp \left( - \frac{|y|^2 + |\beta|^2 \sigma_1^4 - 2|y||\beta| \sigma_1^2 \cos(\phi - \angle y)}{\sigma_1^2} \right) d\phi
\]

\[
= \frac{1}{\pi \sigma_1^2} \exp \left( - \frac{|y|^2 + |\beta|^2 \sigma_1^4}{\sigma_1^2} \right) \cdot \frac{1}{2\pi} \int_0^{2\pi} \exp \left( 2|y||\beta| \cos(\phi - \angle y) \right) d\phi
\]

\[
= \frac{1}{\pi \sigma_1^2} \exp \left( - \frac{|y|^2 + |\beta|^2 \sigma_1^4}{\sigma_1^2} \right) I_0 \left( 2|y||\beta| \right)
\]

Now, to derive the PDF of |\(Y|\), we need to consider the following 2-D transformation of r.v.'s (i.e. from Cartesian to polar coordinates):

\[
\begin{align*}
R &= |Y| = \sqrt{Y_R^2 + Y_I^2} \\
\theta &= \angle Y = \arctan \left( \frac{Y_I}{Y_R} \right)
\end{align*}
\]
From the well-known theorem of r.v. transformations:

\[ p_{y_1, y_2|H_1}(y_1, y_2|H_1) \equiv p_{R, \vartheta|H_1}(r, \vartheta|H_1) = \frac{p_{y_1, y_2|H_1}(y_R, y_I|H_1)}{|J|} \]

Where \( J \) is the Jacobian of the transformation: \( |J| = 1/r \)

\[ p_{R, \vartheta|H_1}(r, \vartheta|H_1) = \frac{r}{\pi \sigma_1^2} \exp \left( - \frac{r^2 + |\beta|^2 \sigma_1^4}{\sigma_1^2} \right) I_0 \left( 2r |\beta| \right), \quad 0 \leq \vartheta < 2\pi, \quad r \geq 0 \]

\[ p_R|H_1(r|H_1) = \int_0^{2\pi} p_{R, \vartheta|H_1}(r, \vartheta|H_1) d\vartheta \]

\[ = \frac{2r}{\sigma_1^2} \exp \left( - \frac{r^2 + |\beta|^2 \sigma_1^4}{\sigma_1^2} \right) I_0 \left( 2r |\beta| \right), \quad r \geq 0 \]
The PDF of the envelope $R = |Y|$ under the hypothesis $H_1$ is a Rician function:

$$p_{R|H_1}(r|H_1) = \frac{2r}{\sigma_1^2} \exp\left(-\frac{r^2 + |\beta|^2 \sigma_1^4}{\sigma_1^2}\right) I_0\left(2r|\beta|\right), \quad r \geq 0,$$

$$SINR_{WMF} = |\beta|^2 \sigma_1^2 = |\beta|^2 \mathbf{v}^H \mathbf{R}^{-1} \mathbf{v}$$

Under the hypothesis $H_0$, the PDF is a Rayleigh function; it can be obtained from the Rice PDF by setting $\beta = 0$:

$$p_{R|H_0}(r|H_0) = \frac{2r}{\sigma_1^2} \exp\left(-\frac{r^2}{\sigma_1^2}\right), \quad r \geq 0, \quad \sigma_1^2 = \mathbf{v}^H \mathbf{R}^{-1} \mathbf{v}$$
\[ z \triangleq \frac{\sqrt{2} r}{\sigma_1}, \quad \gamma \triangleq \sqrt{2 \text{SINR}_{\text{WMF}}}, \quad \text{SINR}_{\text{WMF}} = |\beta|^2 \mathbf{v}^H \mathbf{R}^{-1} \mathbf{v} \]
Performance of the non-coherent WMF – SW0

\[
p_{R|H_1}(r|H_1) = \frac{2r}{\sigma_1^2} \exp \left( -\frac{r^2 + |\beta|^2 \sigma_1^4}{\sigma_1^2} \right) I_0(2r|\beta|), \quad r \geq 0
\]

\[
P_D = \int_{\sqrt{\eta}}^{+\infty} p_{R|H_1}(r|H_1) dr = \int_{\sqrt{\eta}}^{+\infty} \frac{2r}{\sigma_1^2} \exp \left( -\frac{r^2 + |\beta|^2 \sigma_1^4}{\sigma_1^2} \right) I_0(2r|\beta|) dr
\]

\[
= \int_{\lambda}^{+\infty} z \exp \left( -\frac{z^2 + \gamma^2}{2} \right) I_0(z\gamma) dz = Q_M(\gamma, \lambda)
\]

where we defined
\[
z \triangleq \frac{\sqrt{2} r}{\sigma_1}, \quad \gamma \triangleq \sqrt{2} |\beta| \sigma_1 \quad \Rightarrow \quad z \gamma = 2r|\beta|, \quad \lambda = \sqrt{2\eta/\sigma_1^2}
\]

\[Q_M(\gamma, \lambda)\] is the so-called \textbf{Marcum Q-function}
Performance of the non-coherent WMF – SW0

Summarizing:

\[ P_{FA} = \exp \left( -\frac{\eta}{v^H R^{-1} v} \right) \]

Note that: \( SINR_{WMF} = \frac{SINR_{CWMF}}{2} \)

\[ P_D = Q_M \left( \sqrt{2|\beta|^2 v^H R^{-1} v}, \sqrt{\frac{2\eta}{v^H R^{-1} v}} \right) = Q_M \left( \sqrt{2SINR_{WMF}}, \sqrt{2 \ln(1/P_{FA})} \right) \]

Sometimes, \( P_D \) is instead expressed as follows:

\[ P_D = \sum_{n=0}^{+\infty} \left( \frac{\xi^n e^{-\xi}}{n!} \right) \cdot \Gamma \left( n+1, \frac{\eta}{v^H R^{-1} v} \right) \]

where \( \Gamma(.) \) is the incomplete gamma function

and \( \xi \) is the WMF output SINR: \( \xi = SINR_{WMF} = |\beta|^2 v^H R^{-1} v \)
Performance of the non-coherent WMF – SW0

\[ Q_M(\gamma, \lambda) \]

\[ \gamma^2 / 2 = \text{SINR}_{WMF} \]
ROC: the noncoherent WMF (SW-0) vs. coherent WMF.
Performance of the WMF

ROC for narrowband detector: noncoherent WMF (SW-0) vs. coherent WMF.

\[
SINR_{WMF} = \frac{SINR_{CWMF}}{2} = |\beta|^2 v^H R^{-1} v
\]

\[SINR_{WMF}[dB]\]

coherent WMF (blu)
noncoherent WMF (red)

CHvsNCH.m