University of Pisa – PhD Program in Ingegneria dell'Informazione

# Game Theory and Optimization in Communications and Networking

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# Special classes of static games

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Among **noncooperative static games**, there are special classes of games which are particularly **relevant** to address problems in wireless and communication networks:

- Supermodular games
- Potential games
- Generalized Nash games



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# Supermodular games (1/2)

A **supermodular** game is a static noncooperative game, which has:

(1) a set of players $\mathcal{K}$ = [1, …, *K*]

(2) $\mathcal{A}_k$ is the set of strategies (actions) available to player *k*

(3) $u_k(a)$ is the utility (payoff) for player *k*

where, for any $a'_k \geq a_k$ and for any $\mathbf{a}'_{\backslash k} \geq \mathbf{a}_{\backslash k}$ (component-wise),

$$u_k([a'_k, \mathbf{a}'_{\backslash k}]) - u_k([a_k, \mathbf{a}'_{\backslash k}]) \geq u_k([a'_k, \mathbf{a}_{\backslash k}]) - u_k([a_k, \mathbf{a}_{\backslash k}])$$

The utility function has **increasing differences**: if a player takes a higher action, then the other players are better off when taking higher actions as well

# Supermodular games (2/2)

Checking that a game is **supermodular** is **attractive** because:

The set of pure-strategy Nash equilibria is **not empty**

The best-response dynamics **converges** to the smallest element-wise vector in the set of Nash equilibria



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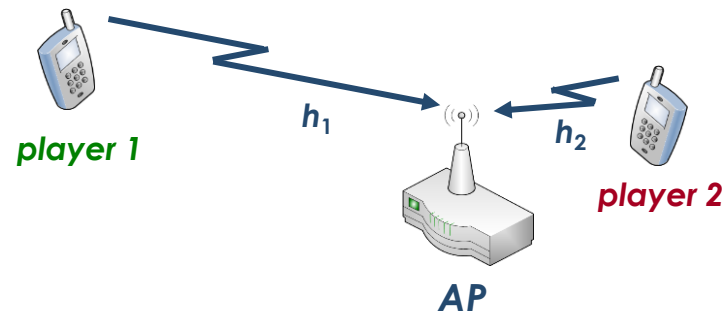
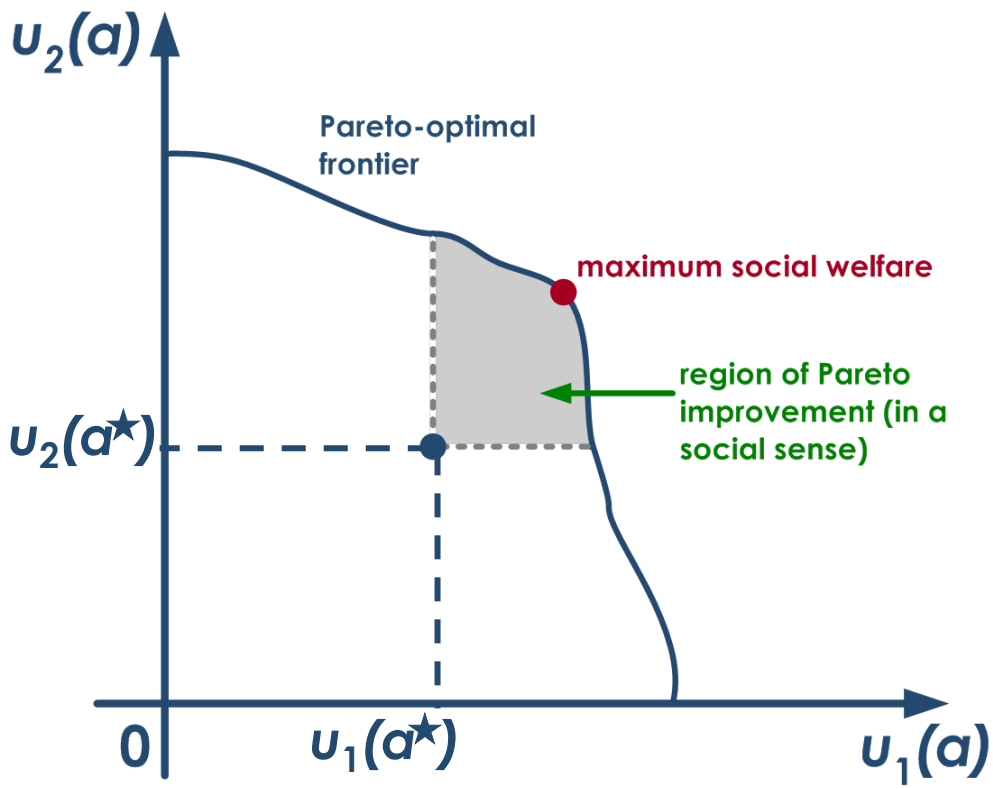
# Supermodular games: An example [24] (1/6)

Let's consider again the near-far power control game with continuous power sets $\mathcal{A}_k=[0, p]$:




The inefficiency of the Nash equilibrium occurs because each terminal ignores the interference it generates to the other

# Supermodular games: An example (2/6)



**Pricing** (a.k.a. **taxation**) is an effective tool used in microeconomics to deal with this problem.

We can in fact take advantage of a **modified** utility:

$$\tilde{u}_k(\mathbf{a}) = u_k(\mathbf{a}) - \alpha \cdot a_k = R \cdot \frac{f(\gamma_k)}{a_k} - \alpha \cdot a_k$$

*pricing factor*

where the pricing factor $\alpha$ is used to introduce some form of **externality**, that charges players proportionally to the powers they radiate (that causes **multiple-access interference**)

# Supermodular games: An example (3/6)

Unlike the game without pricing ($\alpha$ = 0), the utility function is **not** quasi-concave in $a_k$

However, $\tilde{u}_k(\mathbf{a})$ has increasing differences, and hence the game is **supermodular**. To check it, it is **sufficient** to prove that $\tilde{u}_k(\mathbf{a})$ is twice differentiable and such that

$$\frac{\partial^2 \tilde{u}_k(\mathbf{a})}{\partial a_k \partial a_j} \geq 0 \quad \forall k, j \in \mathcal{K}, k \neq j$$

**Exercise 1:** Compute $\partial^2 \tilde{u}_k(\mathbf{a}) / \partial a_k \partial a_j$ in the case $u_k(\mathbf{a}) = Rf(\gamma_k)/a_k$

solution: $\dfrac{\partial^2 \tilde{u}_k(\mathbf{a})}{\partial a_k \partial a_j} = -R \cdot \dfrac{h_j a_j}{M h_k a_k^3} \cdot \dfrac{\partial^2 f(\gamma_k)}{\partial \gamma_k^2}$

# Supermodular games: An example (4/6)



**In the case** $f(\gamma_k) = [1 - e^{-\gamma_k/2}]^L$ ,

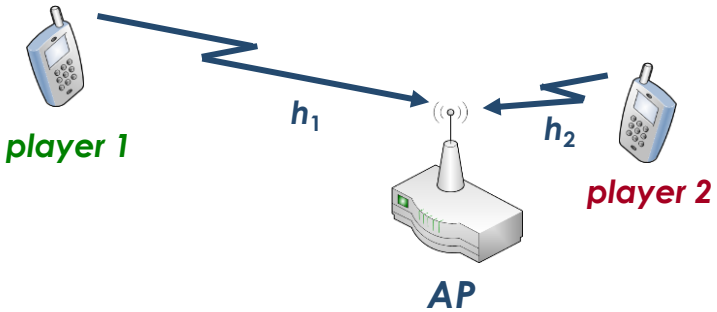
$$\frac{\partial^2 f(\gamma_k)}{\partial \gamma_k^2} = \frac{L}{4} e^{-\gamma_k/2} \cdot [1 - e^{-\gamma_k/2}]^{L-2} \cdot (L e^{-\gamma_k/2} - 1)$$

**and thus**

$$\frac{\partial \tilde{u}_k(\mathbf{a})}{\partial a_k \partial a_j} \geq 0 \quad \Leftrightarrow \quad \gamma_k \geq 2 \log L \quad \forall k \in \mathcal{K}$$

# Supermodular games: An example (5/6)

To include **pricing** in our power-control game, we can modify the definitions of the game ingredients as follows:




(1) $\mathcal{K}$ = [1, 2]

(2) $$\mathcal{A}_k = \left\{ a_k \in [0, \overline{p}] : \gamma_k(\mathbf{a}) = \frac{M h_k a_k}{\sigma^2 + h_{\backslash k} a_{\backslash k}} \geq 2 \log L \right\}$$

(3) $$\tilde{u}_k(\mathbf{a}) = u_k(\mathbf{a}) - \alpha \cdot a_k = R \cdot \frac{f(\gamma_k)}{a_k} - \alpha \cdot a_k$$



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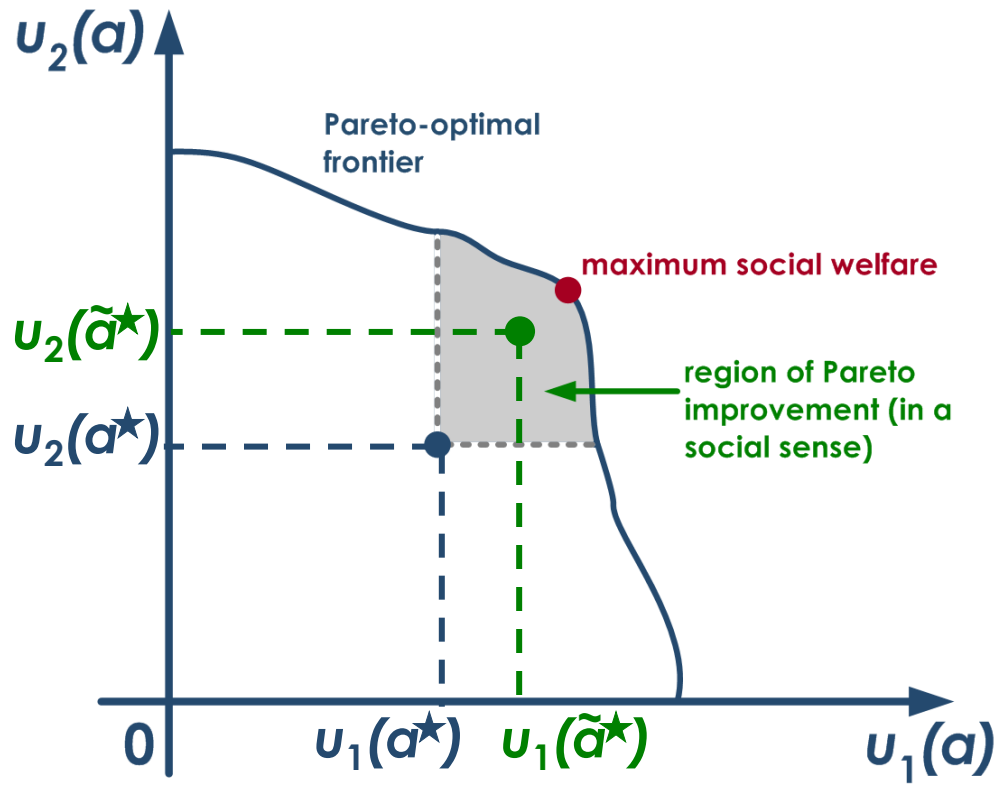
# Supermodular games: An example (6/6)

This modified formulation improves the **efficiency** of the Nash equilibrium (NE):

*$a^\star$: NE of the original game*

*$\tilde{a}^\star$: NE of the pricing game*




Intuitively, this form of externality encourages players to use the resources **more efficiently**

# Potential games (1/2)



A **potential** game is a static noncooperative game, which has:

(1) a set of players $\mathcal{K}$ = [1, …, *K*]

(2) $\mathcal{A}_k$ is the set of strategies *a* (actions) available to player *k*

(3) $u_k(a)$ is the utility (payoff) for player *k*

where, for all players and all other players' profile $a_{\backslash k}$,

$$u_k([a_k, \mathbf{a}_{\backslash k}]) - u_k([a'_k, \mathbf{a}_{\backslash k}]) = \Phi([a_k, \mathbf{a}_{\backslash k}]) - \Phi([a'_k, \mathbf{a}_{\backslash k}]) \quad \forall a_k, a'_k \in \mathcal{A}_k$$

*potential function*

*no dependence on user index!*

(**exact potential game**)

# Potential games (2/2)

A **potential** game is a static noncooperative game, which has:

(1) a set of players $\mathcal{K}$ = [1, …, *K*]

(2) $\mathcal{A}_k$ is the set of strategies (actions) available to player *k*

(3) $u_k(a)$ is the utility (payoff) for player *k*

where, for all players and all other players' profile $a_{\backslash k}$,

$$\mathrm{sgn}[u_k([a_k, \mathbf{a}_{\backslash k}]) - u_k([a'_k, \mathbf{a}_{\backslash k}])] = \mathrm{sgn}[\Phi([a_k, \mathbf{a}_{\backslash k}]) - \Phi([a'_k, \mathbf{a}_{\backslash k}])] \quad \forall a_k, a'_k \in \mathcal{A}_k$$

(**ordinal potential game**)



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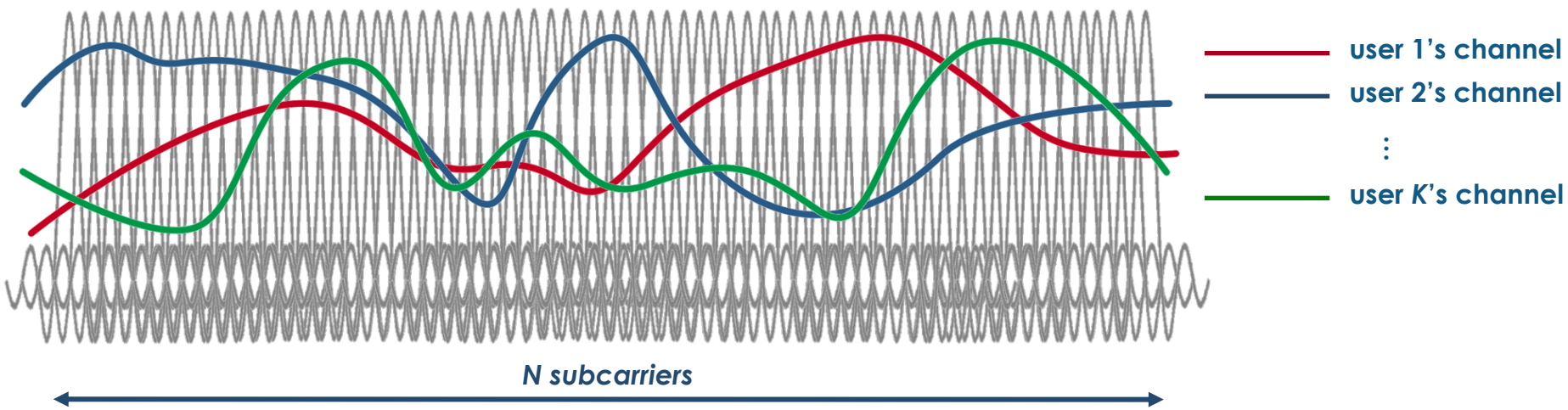
# Potential games: An example [26] (1/5)

Uplink power allocation for an OFDMA cellular network, with *N* subcarriers and *K* terminals




Each terminal *k*'s objective: allocate the power so as to maximize its own achievable rate given a constraint on the maximum total radiated power $\bar{p}_k$ to exploit the channel frequency diversity

# Potential games: An example (2/5)

The strategic-form representation of the game is the following:

(1) **players:** $\mathcal{K}$ = [1, …, *K*] terminals in the network

(2) **strategies:** $\mathcal{A}_k = \left\{ \mathbf{p}_k \in \mathbb{R}^N : p_k(n) \geq 0 \, \forall n, \sum_{n=1}^{N} p_k(n) \leq \overline{p}_k \right\}$

(3) **utilities:** $u_k([\mathbf{p}_k, \mathbf{p}_{\setminus k}]) = \sum_{n=1}^{N} \log(1 + \gamma_k(n))$ *Shannon capacity*

where

$$\gamma_k(n) = \frac{h_k(n) p_k(n)}{\sigma^2 + \sum_{j \neq k} h_j(n) p_j(n)}$$

and $h_k(n)$ are terminal *k*'s SINR and channel power gain over subcarrier *n*, resp.

# Potential games: An example (3/5)

This is an **exact** potential game, with potential function

$$\Phi([\mathbf{p}_k, \mathbf{p}_{\setminus k}]) = \sum_{n=1}^{N} \log \left( \sigma^2 + \sum_{j=1}^{K} h_j(n) p_j(n) \right)$$

**Exercise 2:** Check that $u_k([\mathbf{p}_k, \mathbf{p}_{\setminus k}]) - u_k([\mathbf{p}'_k, \mathbf{p}_{\setminus k}]) = \Phi([\mathbf{p}_k, \mathbf{p}_{\setminus k}]) - \Phi([\mathbf{p}'_k, \mathbf{p}_{\setminus k}])$

# Potential games: An example (4/5)

In an infinite potential game (as occurs in this case), a pure-strategy Nash equilibrium **exists** if:

- the strategy sets $\mathcal{A}_k$ are compact
- the potential function $\Phi$ is upper semi-continuous on $\mathcal{A}=\mathcal{A}_1\times\ldots\times\mathcal{A}_K$

The interest in potential games stems from the **guarantee** of existence of pure-strategy Nash equilibria, and from the study of a **single-variable** function (the potential one)

If $\mathcal{A}$ is a compact and convex set, and $\Phi(a)$ is a continuously diffentiable function and **stricly concave**, then the Nash equilibrium of the potential game is **unique**



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# Potential games: An example (5/5)

The **convergence** to one of the Nash equilibria of the game can be achieved using an iterative algorithm based on the **best-response** criterion:

$$\mathbf{p}_k^{(t+1)} = \arg \max_{\mathbf{p}_k \in \mathbb{R}^N} \Phi\left(\left[\mathbf{p}_k, \mathbf{p}_{\backslash k}^{(t)}\right]\right)$$

$$\text{s.t.} \sum_{n=1}^{N} p_k(n) \leq \overline{p}_k \text{ and } p_k(n) \geq 0, \forall n$$

The solution is given by the well-known **iterative water-filling** algorithm:

$$p_k^{(t+1)}(n) = \max\left(0, \frac{1}{\lambda_k} - \frac{p_k^{(t)}(n)}{\gamma_k^{(t)}(n)}\right)$$

*fed back by the base station*

where $\lambda_k$ is such that

$$\sum_{n=1}^{N} \max\left(0, 1/\lambda_k - p_k^{(t)}(n)/\gamma_k^{(t)}(n)\right) = \overline{p}_k$$

# Generalized Nash games

A **generalized Nash game** is a static noncooperative game, with:

(1) a set of players $\mathcal{K} = [1, \ldots, K]$

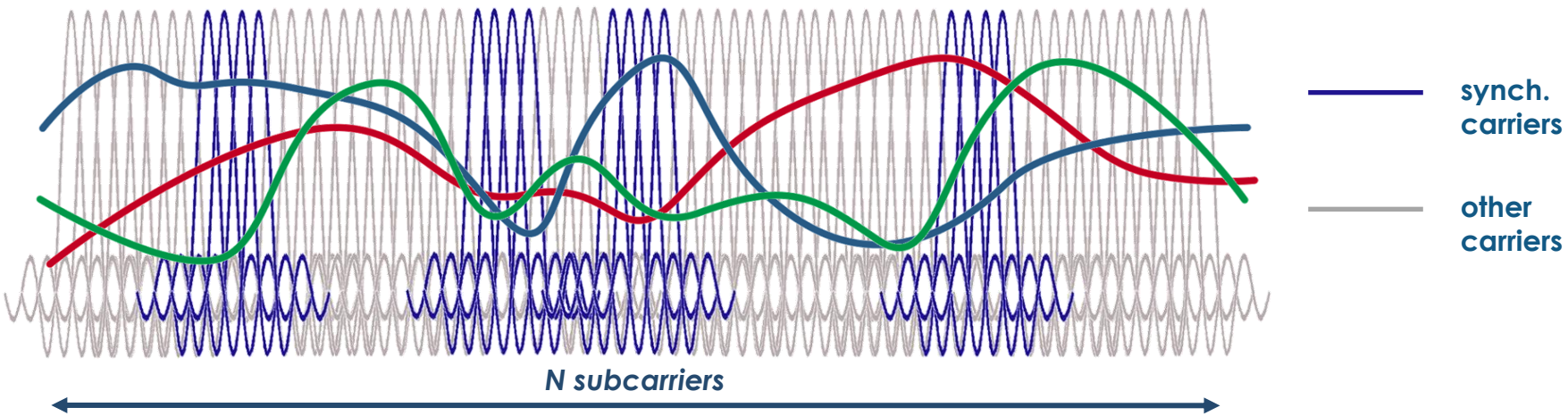
(2) player $k$'s set of strategies $\mathcal{A}_k$ depends on **all others' actions** $a_{\backslash k}$

(3) $u_k(a)$ is the utility (payoff) for player $k$

The **interplay** between strategy sets typically occurs when placing **constraints** on the game formulation

# Generalized Nash games: An example [28] (1/2)

Let's focus again on the uplink of an OFDMA cellular network, this time during the (contention-based) **network association** phase, in which the subcarriers are shared by the terminals in a **multicarrier CDMA** fashion




**Each terminal *k*'s objective:** allocate the power so as to **minimize the energy** spent to get a successful association given quality of service (QoS) constraints in terms of **false code locks**

# Generalized Nash games: An example (2/2)

This situation can be modeled by the following strategic-form representation:

(1) **players:** $\mathcal{K}$ = [1, …, $K$] terminals in the network

(2) **strategies:** power strategy sets $\mathcal{A}_k$=[0, $\bar{p}_k$]

(3) **QoS constraints:** max. probabilty of false alarm

max. mean square error on timing estimation

(4) **utilities:** $u_k(\mathbf{a}) = \frac{\Pi_k(\mathbf{a})}{T \cdot a_k}$

*probability of correct network association*

*energy spent per correct network association*

The solution of generalized Nash games is the **generalized Nash equilibrium**: in this case, it exists and is unique if and only if the number of terminals $K$ is below a **certain threshold**



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# Beyond Nash equilibrium: other equilibrium concepts



In addition to the concept of mixed-strategy and pure-strategy Nash equilibria in noncooperative static games, it is worth mentioning:

- the **correlated equilibrium**: a generalization of the Nash equilibrium, where an arbitrator (not necessarily an intelligent entity) helps the players to correlate their actions, so as to favor a decision process **in between** non-cooperation and cooperation

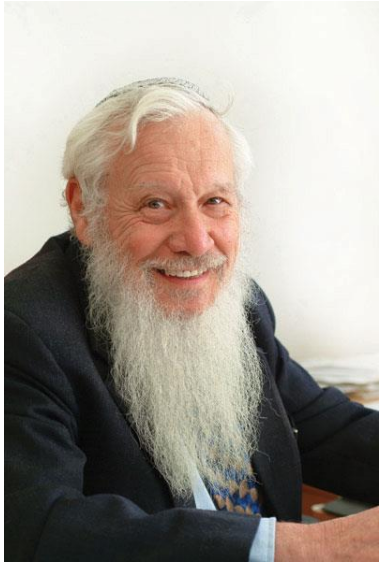
  **example:** it may select one of the two pure-strategy Nash equilibria in the multiple-access game

- the **Wardrop equilibrium**: a limiting case of the Nash equilibrium when the population of users becomes **infinite**

  **example:** it can be used as a decision concept to select routing strategies in ad-hoc networks

# Correlated equilibrium (1/3)

The **correlated equilibrium** is a generalization of the Nash equilibrium, in which an arbitrator can generate some random signals (according to a mechanism known by the players) that can help them to **coordinate** their actions, so as to **improve** the performance in a social sense



*Robert J. Aumann, 1930-*

**(DEF:) A probability distribution $\pi(\mathbf{a}) = \{\pi(\mathbf{a})\}_{\mathbf{a}\in\mathcal{A}}$ is a correlated equilibrium of game $\mathcal{G}$ if, for all $k \in \mathcal{K}$, $a_k, a'_k \in \mathcal{A}_k$, and $\mathbf{a}_{\setminus k} \in \mathcal{A}_{\setminus k}$,**

$$\sum_{\mathbf{a}_{\setminus k}\in\mathcal{A}_{\setminus \mathbf{k}}} \pi(a_k, \mathbf{a}_{\setminus k}) \cdot \left[u_k(a_k, \mathbf{a}_{\setminus k}) - u_k(a'_k, \mathbf{a}_{\setminus k})\right] \geq 0$$



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# Correlated equilibrium (2/3)

Let's consider the **multiple access game** again:

- $K=2$, $\mathcal{A}_k = \{w, t\}$
- $\mathcal{A} = \{ (w,w), (w,t), (t,w), (t, t) \}$

| player 1 \ player 2 | wait | transmit |
|---|---|---|
| wait | 0, 0 | 0, $t-c$ |
| transmit | $t-c$ , 0 | $-c$, $-c$ |

$$\pi_{(\mathrm{w,w})} \cdot [0-(t-c)] + \pi_{(\mathrm{w,t})} \cdot [0-(-c)] \geq 0$$
$$\pi_{(\mathrm{t,w})} \cdot [(t-c)-0] + \pi_{(\mathrm{t,t})} \cdot [(-c)-0] \geq 0$$
$$\pi_{(\mathrm{w,w})} \cdot [0-(t-c)] + \pi_{(\mathrm{t,w})} \cdot [0-(-c)] \geq 0$$
$$\pi_{(\mathrm{w,t})} \cdot [(t-c)-0] + \pi_{(\mathrm{t,t})} \cdot [(-c)-0] \geq 0$$
$$\pi_{(\mathrm{w,w})} + \pi_{(\mathrm{w,t})} + \pi_{(\mathrm{t,w})} + \pi_{(\mathrm{t,t})} = 1$$
$$\pi_{(\mathrm{w,w})} \geq 0, \pi_{(\mathrm{w,t})} \geq 0, \pi_{(\mathrm{t,w})} \geq 0, \pi_{(\mathrm{t,t})} \geq 0$$

# Correlated equilibrium (3/3)

This system of inequalities adminits an **infinite** number of solutions (i.e., correlated equilibria). **Notable** solutions are:

- $\pi_{(\mathrm{w,w})} = 0, \pi_{(\mathrm{w,t})} = 1, \pi_{(\mathrm{t,w})} = 0, \pi_{(\mathrm{t,t})} = 0$
- $\pi_{(\mathrm{w,w})} = 0, \pi_{(\mathrm{w,t})} = 0, \pi_{(\mathrm{t,w})} = 1, \pi_{(\mathrm{t,t})} = 0$

} pure-strategy Nash equilibria

- $\pi_{(\mathrm{w,w})} = c^2/t^2, \pi_{(\mathrm{w,t})} = c/t\,(1 - c/t)\,,$
  $\pi_{(\mathrm{t,w})} = c/t\,(1 - c/t)\,, \pi_{(\mathrm{t,t})} = (1 - c/t)^2$

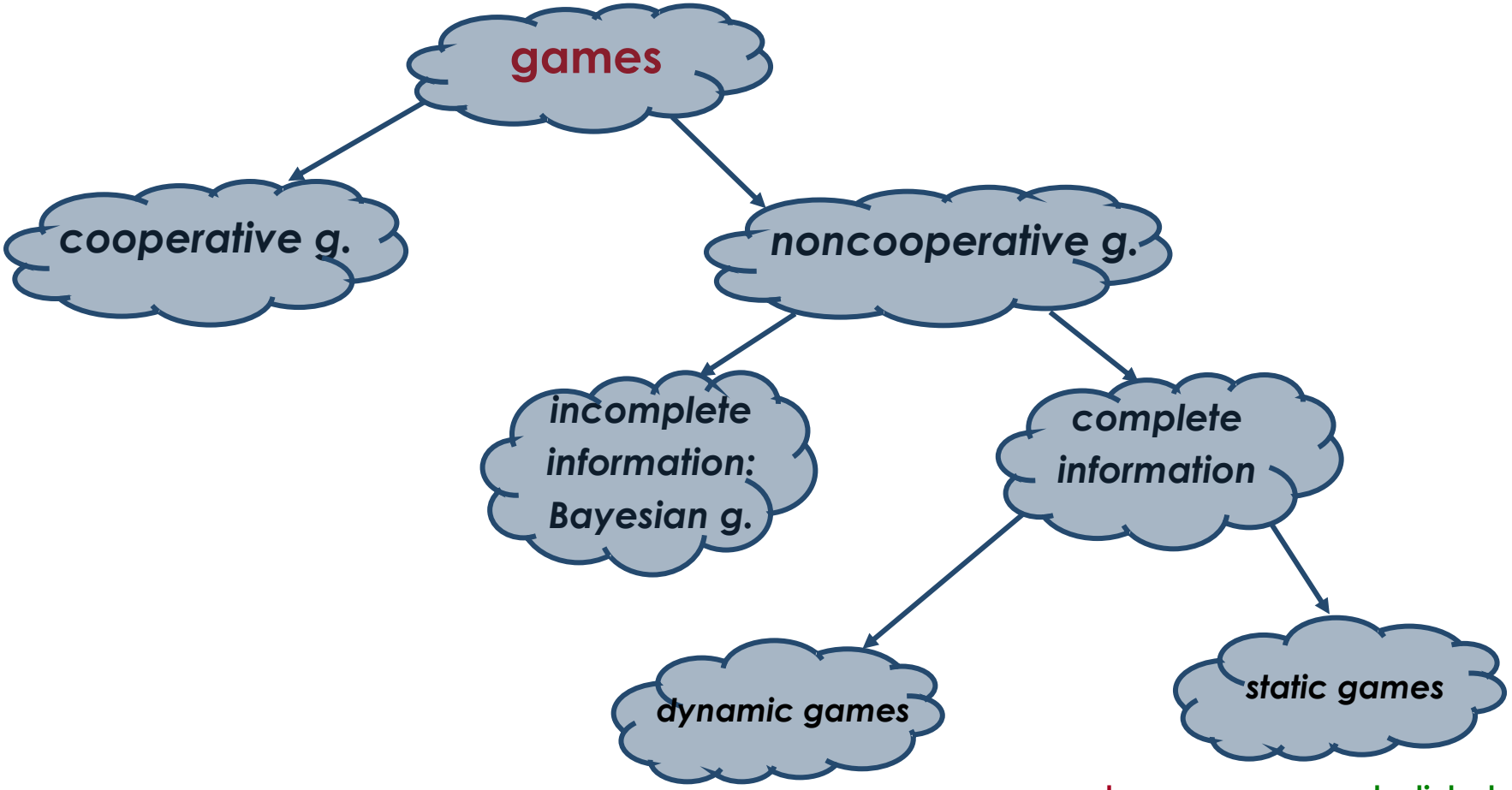
mixed-strategy Nash equilibrium

- $\pi_{(\mathrm{w,w})} = 0, \pi_{(\mathrm{w,t})} = 1/2, \pi_{(\mathrm{t,w})} = 1/2, \pi_{(\mathrm{t,t})} = 0$
- $\pi_{(\mathrm{w,w})} = \frac{c}{t-2c}, \pi_{(\mathrm{w,t})} = \pi_{(\mathrm{t,w})} = \frac{t-c}{t-2c}, \pi_{(\mathrm{t,t})} = 0$

**Exercise 2:** Check that the fourth solution maximizes both the individual expected payoffs and the expected sum-utility

# Lunch Time !

# Taxonomy revisited




classes: zero-sum, potential, etc.
tools: iterated dominance, mixed strategies, etc.
solution: Nash e., correlated e., etc.

# Dynamic games

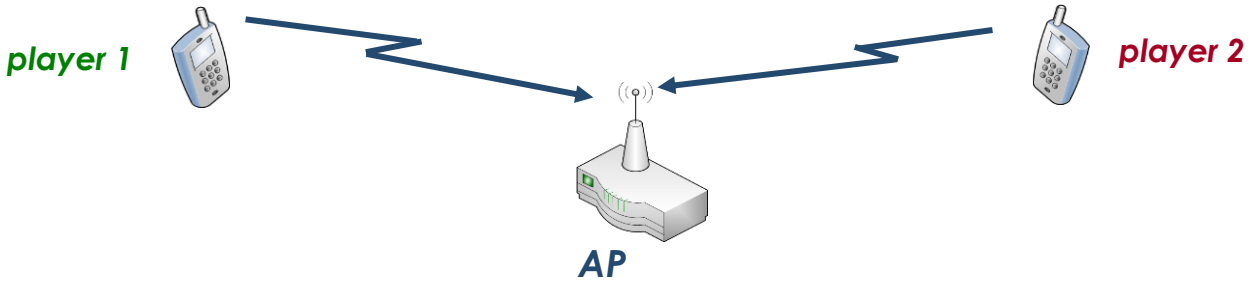
# Dynamic games

There could be situations in which the players are allowed to have a **sequential** interaction, meaning that the move of one player is conditioned by the previous moves in the game: This is the class of **dynamic games**

Consider a **sequential multiple-access game**, where the packet transmission is successful when there is no collision, in which:

(1) player 1 can either transmit (with a transmission cost c) or wait;

(2) then, **after observing player 1's action**, player 2 chooses whether to transmit or not



# Extensive-form representation (1/2)

A convenient way to describe a sequential game such as the sequential multiple-access game is the **extensive-form representation**, which consists of:

(1) a set of *players*;

(2a) the *order* of moves – i.e., who moves when;

(2b) what the players' *choices* are when they move;

(2c) the *information* each player has when he/she makes his/her choices;

(3) the *payoff* received by each player for each combination of strategies that could be chosen by the players.



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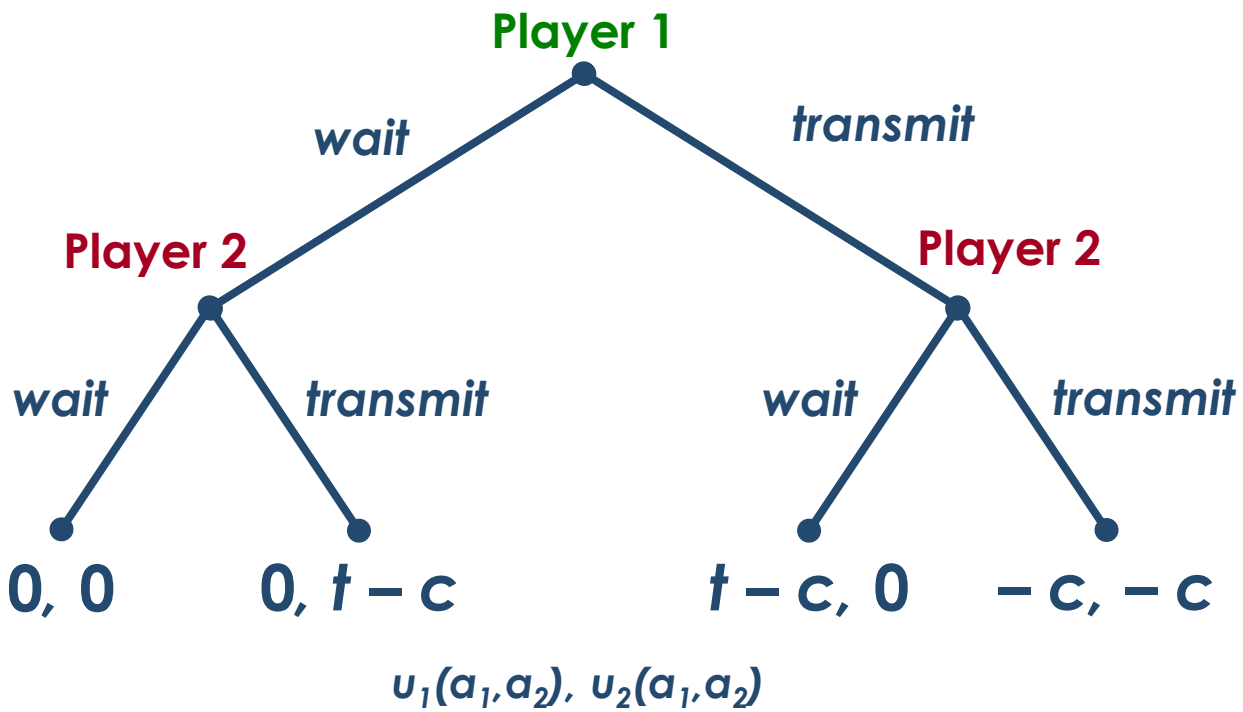
# Extensive-form representation (2/2)

The extensive-form representation of the sequential multiple-access game [29] is




- this is a game with **perfect information**, because the player with the move knows the full history of the play of the game thus far;
- this is a game with **complete information**, because the $u_k(a)$'s are common knowledge;
- this game is a **finite-horizon game**, because the number of stages is finite.

# Nash equilibrium

Any game (both static and dynamic) can **indifferently** be described by its strategic-form representation and its extensive-form representation. The options (the **strategy**) we show for player 2 are his/her response conditioned on what player 1 does (wait/transmit), i.e., a plan for **every** history of the game

**Player 2 (follower)**

| Player 1 (leader) | *W W* | *W T* | *T W* | *T T* |
|---|---|---|---|---|
| *W* | 0, 0 | 0, 0 | 0, $t-c$ | 0, $t-c$ |
| *T* | $t-c$, 0 | $-c$, $-c$ | $t-c$, 0 | $-c$, $-c$ |

There are three **Nash equilibria**: (*T*, (*W*,*W*)), (*T*,(*T*,*W*)), and (*W*,(*T*,*T*))



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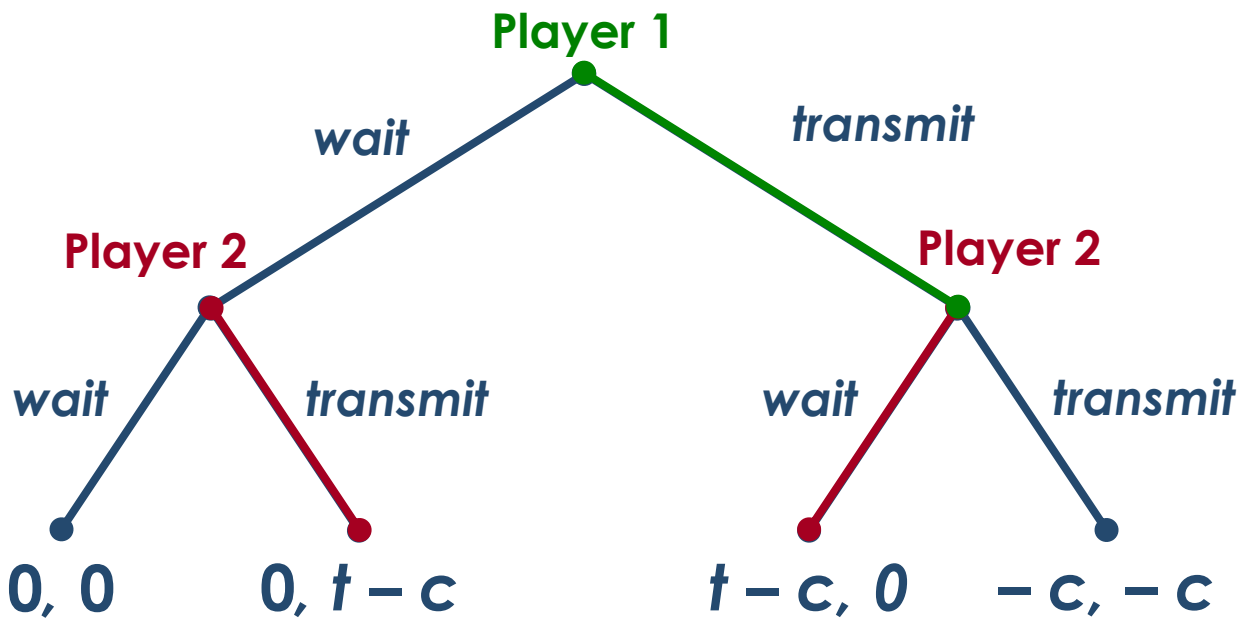
# Backward induction & subgame-perfect Nash equilibrium

Let's go back to the extensive-form representation:




Are the three Nash equilibria, (*T*, (*W*,W)), (*T*, (*T*,*W*)), and (*W*, (*T*,*T*)), all **credible**?

To answer this question, let's resort to the **backward induction** of the game

The strategy (*T*, (*T*,*W*)) is the only **subgame-perfect Nash equilibrium**

# Subgame-perfect Nash equilibrium: A summary

A **subgame-perfect Nash equilibrium** is a strategy profile which prescribes actions that are optimal at each game stage for **every history** of the game, i.e., for any **possible** unfolding of the game

Otherwise stated, a subgame-perfect Nash equilibrium is a Nash equilibrium of **any proper subgame** of the original game (i.e., it is a **credible** Nash equilibrium that has **survived** backward induction

**Some implications:**

- any finite game with complete information has a subgame-perfect Nash equilibrium, perhaps in mixed strategies;
- any finite game with complete and perfect information has (at least) one pure-strategy subgame-perfect Nash equilibrium;



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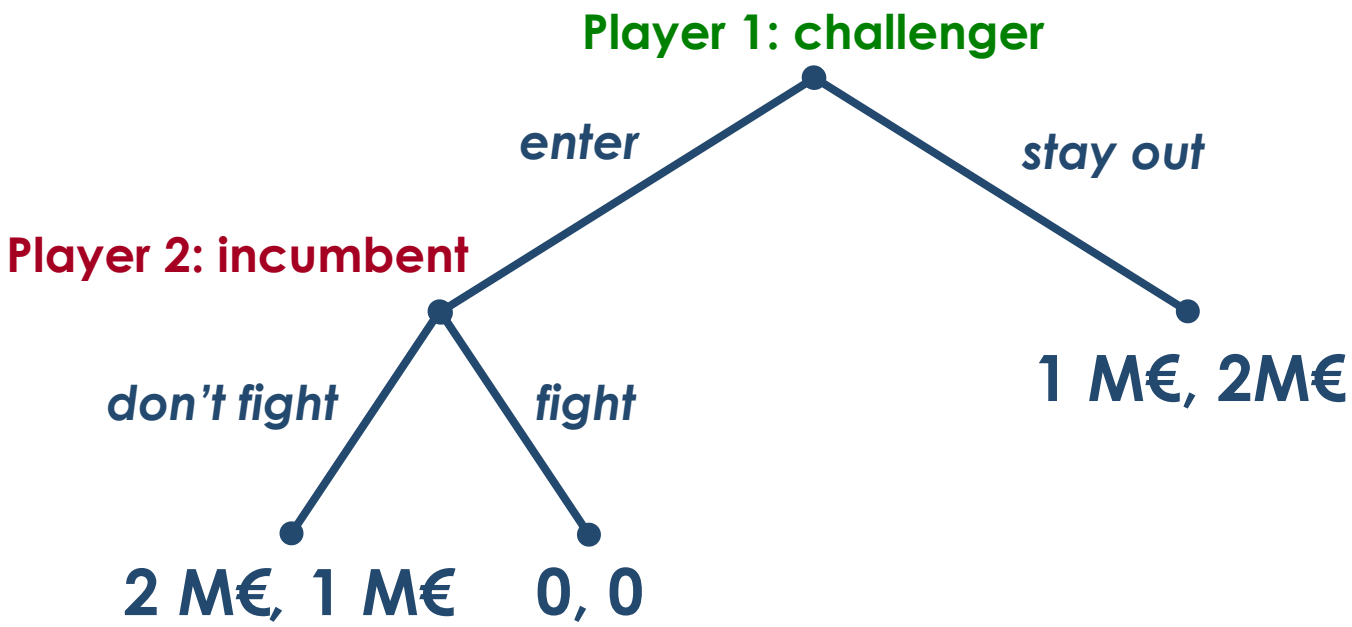
# Example 1

Let's consider the following game, called the **entry game**:



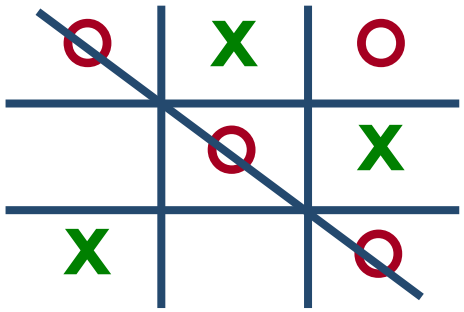

Let's build the **strategic-form** representation of this game

Then, let's identify the Nash equilibria of the game

Finally, using backward induction, let's find the (only) **subgame-perfect NE**

# Tic-Tac-Toe (1/2)

Assume you play **tic-tac-toe**:




In theory, there are:

- $3^9$=19,683 possible board layouts
- 9!=362,880 different sequences of X's and O's on the board

When O is making the **first move**, we have:

- 131,184 games won by O
- 77,904 games won by X
- 46,080 tied games



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# Tic-Tac-Toe (2/2)

Despite its apparent simplicity, it requires some **complex** mathematics to study its outcomes. However, tic-tac-toe can be easily recognized as a **finite** extensive game with **perfect** information

Thus, there exists at least **one** subgame-perfect Nash equilibrium. In this particular game, each player has a subgame perfect equilibrium that guarantees a **tie**, unless the other player deviates from its own: in such case, the equilibrium leads to a **win**.

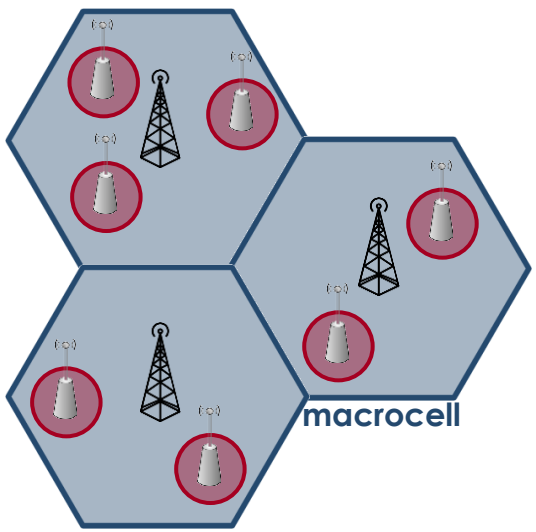
What about **chess**? Chess is **not** a finite game!

However, if you assume to declare a tie once a position is repeated a finite number (e.g., 3) of times, then you can model it as a finite extensive game with perfect information (with a **huge complexity**!)

# Stackelberg games: An example [32]

Games such as the sequential multiple-access game are a particular class of dynamic games: **Stackelberg games**, that have a **leader** and a **follower**

Stackelberg games are particularly suitable to analyze wireless networks: e.g., they can investigate power allocation schemes for **femtocell networks**




**leaders:** macrocell base stations (BSs)

**followers:** femtocell access points (FAPs)

**each player's objective:** allocate the power so as to maximize the achievable rate according to the Shannon capacity formula

This multi-leader/multi-follower formulation leads to **multiple** Stackelberg equilibria

# Dynamic games with imperfect information

**Imperfect information** occurs when, at some stage(s) of the game, some players do **not** know the full history of the play thus far

**Example:** a situation in which the two terminals play the sequential multiple-access game first, and then the (simultaneous) multiple-access game in case of a collision during the first stage, is a game with imperfect information

**Static** games are a special case of dynamic games with imperfect information

| Complete information | ≠ | Perfect information |
|---|---|---|
| (knowledge of the structure of the game: players, strategies, utilities, etc.) | | (knowledge of the full history of the game) |



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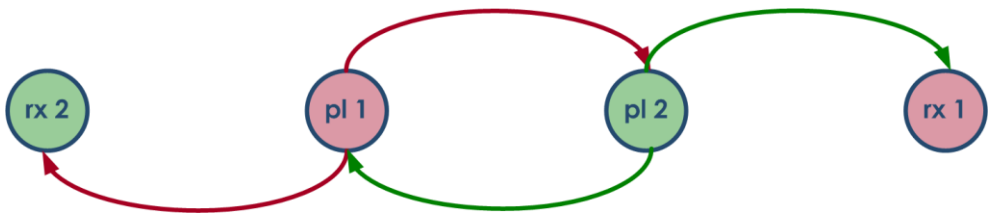
# Repeated games (1/4)

**Repeated games** are a subclass of dynamic games, in which the players face the **same** single-stage (static) game every period

Suppose the forwarder's dilemma is played ***N* times** [33], with $N < \infty$



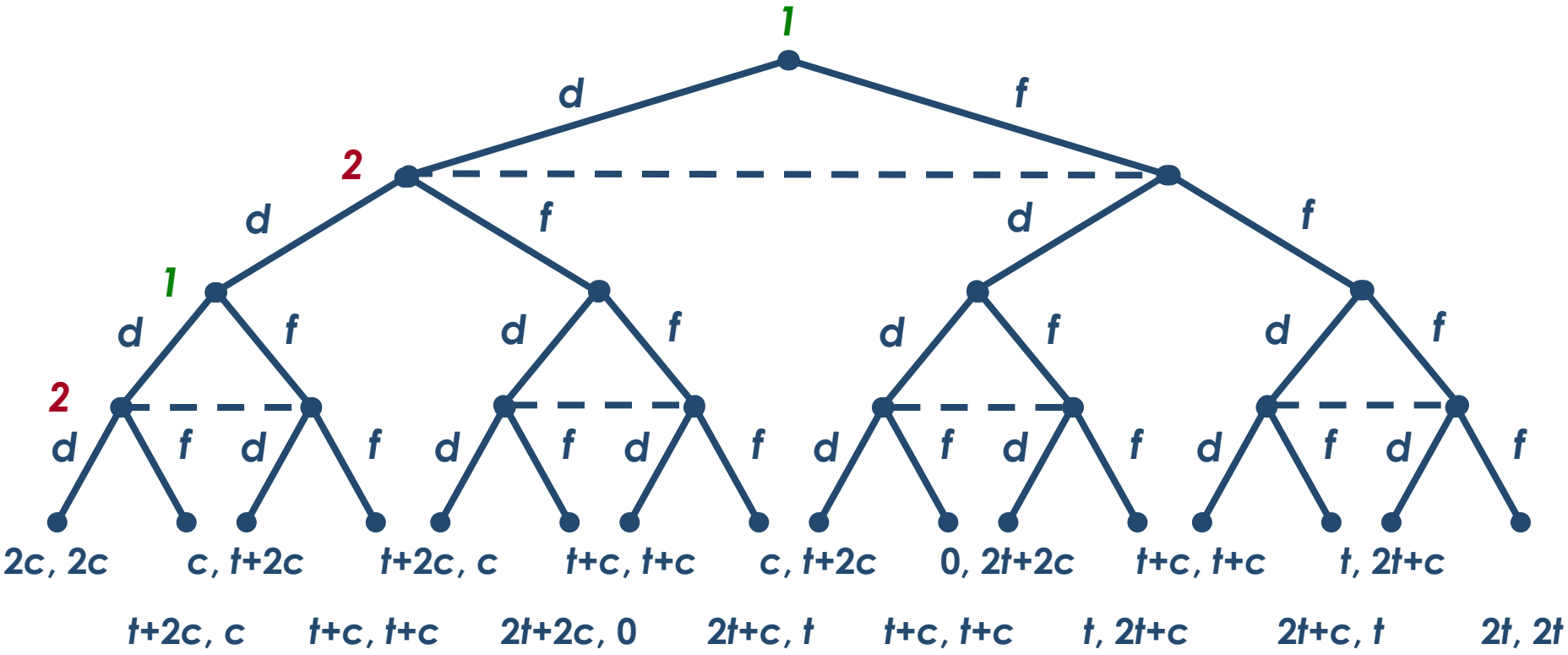

Both players move **simultaneously** after knowing all previous actions

| | | player 2: drop | player 2: forward |
|---|---|---|---|
| player 1 | drop | $c, c$ | $t+c, 0$ |
| player 1 | forward | $0, t+c$ | $t, t$ |

$u_1(a_1,a_2),\ u_2(a_1,a_2)$

# Repeated games (2/4)

Suppose that the game is played **twice** (*N*=2). Its **extensive**-form representation is given by the following tree:



# Repeated games (3/4)

To find the Nash equilibria of this two-stage game, we can build its **equivalent** strategic-form representation:

| player 1 \ player 2 | (d, d) | (d, f) | (f, d) | (f, f) |
|---|---|---|---|---|
| (d, d) | 2c, 2c | t+2c, c | t+2c, c | 2t+2c, 0 |
| (d, f) | c, t+2c | t+c, t+c | t+c, t+c | 2t+c, t |
| (f, d) | c, t+2c | t+c, t+c | t+c, t+c | 2t+c, t |
| (f, f) | 0, 2t+2c | t, 2t+c | t, 2t+c | 2t, 2t |

The **unique** Nash equilibrium of the game is ( (d,d), (d,d) )

**Exercise 3:** Check that it is also subgame-perfect

# Repeated games (4/4)

Using backward induction, both players will end up dropping the packets at every stage, and **no cooperation** is enforced: since the stage game has a unique Nash equilibrium, it is played at every stage

This is true in general: if the stage game $\mathcal{G}$ has a unique Nash equilibrium, then the **finitely-repeated** game $\mathcal{G}_N$ (i.e., with $N < \infty$) has a **unique** subgame-perfect Nash equilibrium, given by playing the Nash equilibrium of $\mathcal{G}$ at **every stage**



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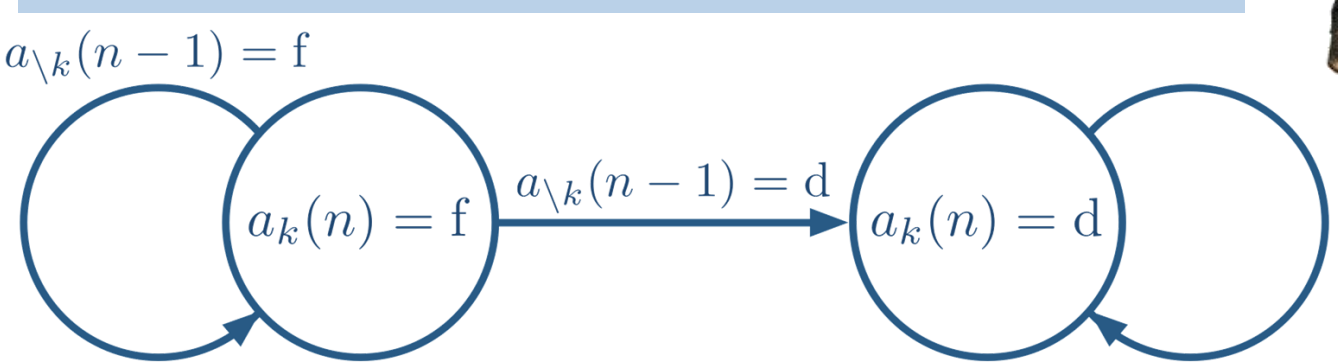
# Infinite-horizon repeated games (1/5)

What if the stage game is repeated an infinite number of times ($N = \infty$)?

In this case (**infinite-horizon** game), a **free rider** behavior may be punished in subsequent rounds

Consider the following strategy (known as **grim-trigger strategy**):

- Keep playing the **social-optimal** solution unless the other player has defected: in this case, **switch** to the noncooperative (selfish) solution



# Infinite-horizon repeated games (2/5)

Implicitly, we have derived the outcomes of the game $\mathcal{G}_N$ by **summing up** each stage's payoffs

We can introduce a **discount factor** $\delta$ ($0 \leq \delta \leq 1$) that measures the **patience** of the players:

$$u_k^{\delta}(\mathbf{a}) = (1-\delta)\sum_{n=0}^{\infty}\delta^n u_k(\mathbf{a}(n))$$

- the smaller $\delta$, the less important future payoffs are (players are **impatient**): myopic (**short-sighted**) optimization
- when $\delta$ = 1, €1 earned today is **equivalent** to €1 earned in 2020: **long-sighted** optimization

# Infinite-horizon repeated games (3/5)

Suppose $\delta$ is given and common to both players, and player 2 uses the grim-trigger strategy $\mathbf{a}_2^{\mathrm{GT}} = \{a_2(n)\}_{n=0}^{\infty}$, with

$$a_2(n) = \begin{cases} \mathrm{f}, & \text{if } a_1(n-1) = a_2(n-1) = \mathrm{f} \\ \mathrm{d}, & \text{otherwise} \end{cases}$$

If player 1 plays 'f' every time, i.e., $\mathbf{a}_1 = \mathbf{a}_1^f = \{a_1(n)\}_{n=0}^{\infty}$, $a_1(n) = \mathrm{f}\ \forall n$ :

$$u_1^{\delta}(\mathbf{a}_1^f, \mathbf{a}_2^{\mathrm{GT}}) = (1-\delta)\sum_{n=0}^{\infty} \delta^n \cdot u_1(\mathrm{f},\mathrm{f}) = t$$

If player 1 plays 'd' every time, i.e., $\mathbf{a}_1 = \mathbf{a}_1^d = \{a_1(n)\}_{n=0}^{\infty}$, $a_1(n) = \mathrm{d}\ \forall n$ :

$$u_1^{\delta}(\mathbf{a}_1^d, \mathbf{a}_2^{\mathrm{GT}}) = (1-\delta)\sum_{n=0}^{\infty} \delta^n \cdot u_1(a_1(n), a_2(n))$$

$$= (1-\delta)\cdot\big((t+c) + c\delta + c\delta^2 + \dots\big) = (1-\delta)t + c$$

**Exercise 4:** Check that $u_1^{\delta}(\mathbf{a}_1, \mathbf{a}_2^{\mathrm{GT}}) < u_1^{\delta}(\mathbf{a}_1^d, \mathbf{a}_2^{\mathrm{GT}})$ for $\mathbf{a}_1 \neq \mathbf{a}_1^d, \mathbf{a}_1^f$

# Infinite-horizon repeated games (4/5)

What is the best strategy for player 1? To answer this question, we must **compare** the two expected payoffs:

$$t \gtrless (1-\delta)t + c$$

- If $\delta \geq c/t$, i.e., if player 1 is **patient enough**, $\mathrm{a}_1^{\mathrm{GT}}$ is the best strategy for player 1 too
- If $\delta < c/t$, player 1's best strategy is to **defect** at every stage

Due to the **symmetry** of the problem, $\mathrm{a}^\star = (\mathrm{a}_1^{\mathrm{GT}}, \mathrm{a}_2^{\mathrm{GT}})$ is a subgame-perfect Nash equilibrium of the game $\mathcal{G}_\infty$ when $\delta \geq c/t$

In the repeated forwarder's dilemma, **cooperation** is enforced by letting the players interact an infinite (i.e., unknown) number of times: this is due to **threatening future punishments** for those who defect



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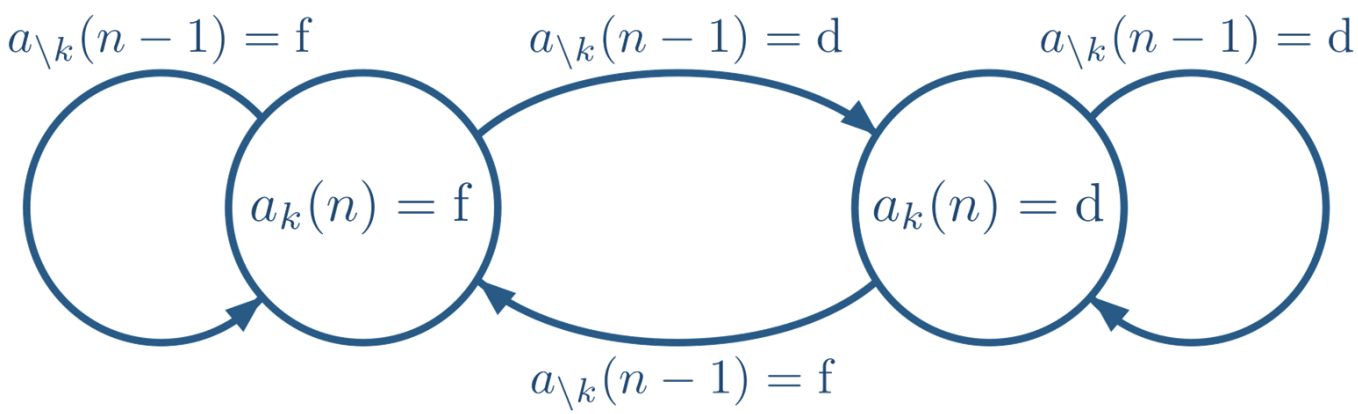
# Infinite-horizon repeated games (5/5)



**Is the grim-trigger strategy the only one that induces cooperation?**

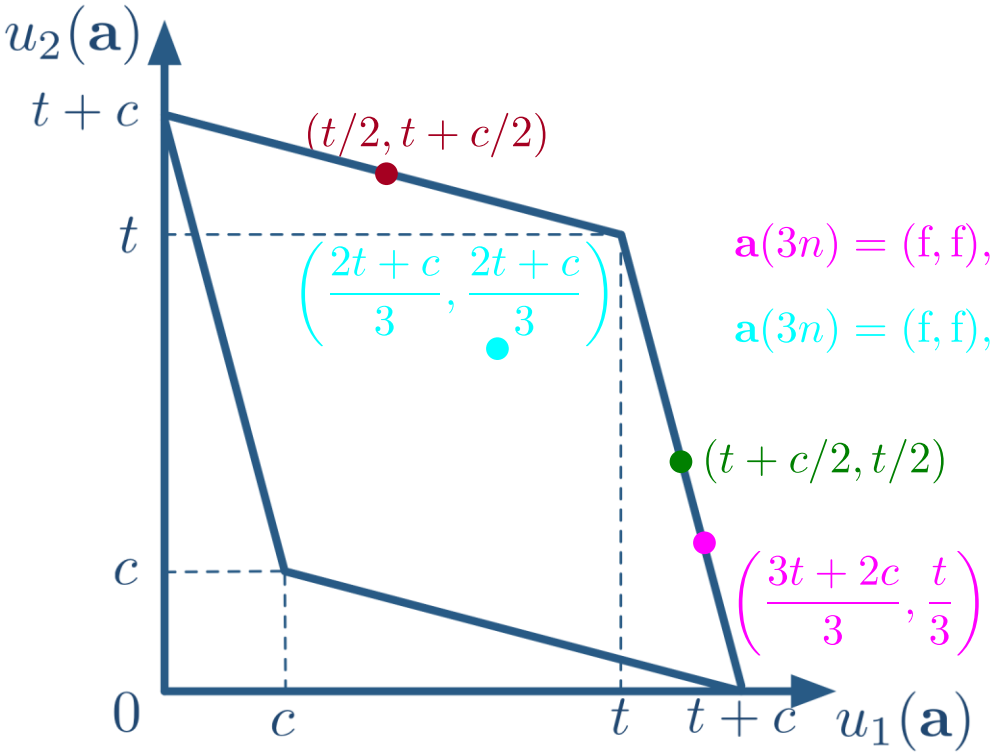
**Excercise 5:** Consider the strategy known as **tit-for-tat**, where

$$a_k(n) = a_{\setminus k}(n-1)$$



# Folk theorem (1/2)

The set of **feasible** discounted payoffs of $\mathcal{G}_\infty$ is any **convex** combination of the pure-strategy payoffs of the constituent game $\mathcal{G}$:




$\mathbf{a}(2n) = (\mathrm{f}, \mathrm{f}), \mathbf{a}(2n+1) = (\mathrm{f}, \mathrm{d}), \delta = 1$

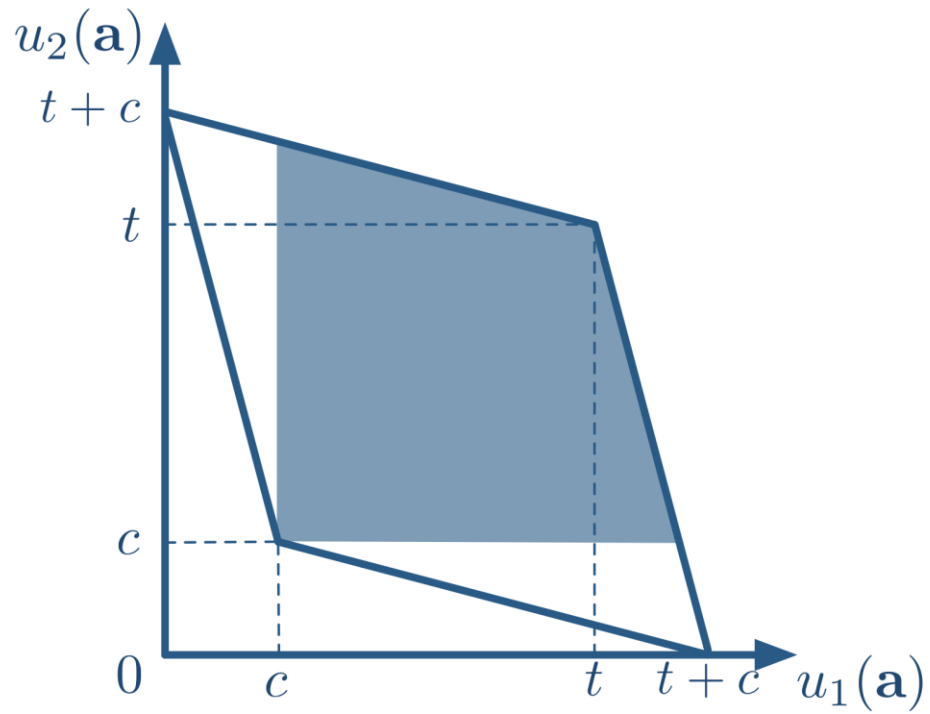
$\mathbf{a}(2n) = (\mathrm{f}, \mathrm{f}), \mathbf{a}(2n+1) = (\mathrm{d}, \mathrm{f}), \delta = 1$

$\mathbf{a}(3n) = (\mathrm{f}, \mathrm{f}), \mathbf{a}(3n+1) = (\mathrm{d}, \mathrm{f}), \mathbf{a}(3n+2) = (\mathrm{d}, \mathrm{f}), \delta = 1$

$\mathbf{a}(3n) = (\mathrm{f}, \mathrm{f}), \mathbf{a}(3n+1) = (\mathrm{d}, \mathrm{f}), \mathbf{a}(3n+2) = (\mathrm{f}, \mathrm{d}), \delta = 1$

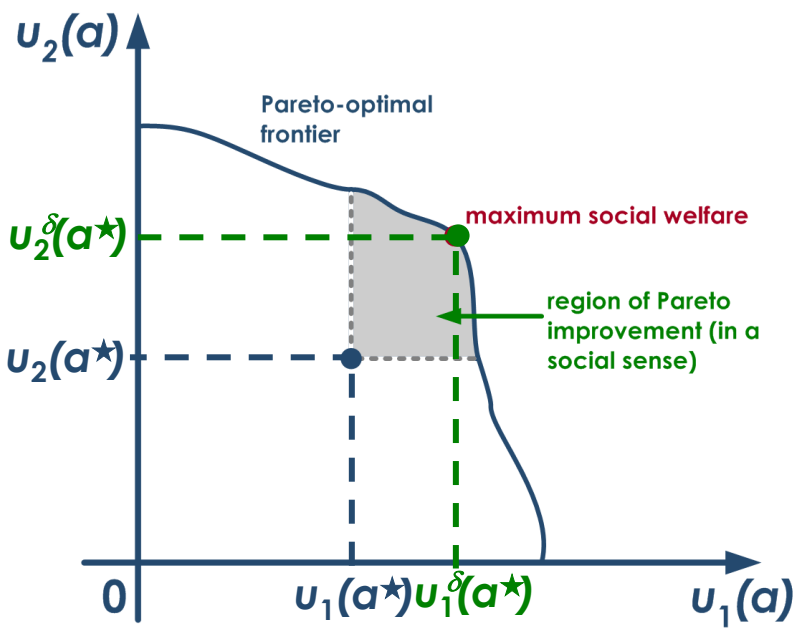
# Folk theorem (2/2)

(Friedman, 1971) Let $\mathcal{G}$ be a finite static game with complete information, with a unique pure-strategy Nash equilibrium $\mathbf{a}^\star$. Let $\{u_k(\mathbf{a}^\star)\}_{k=1}^K$ be the payoffs at the Nash equilibrium. For **any feasible set** $\{\hat{u}_k\}_{k=1}^K$ such that $\hat{u}_k > u_k(\mathbf{a}^\star)$ for all *k*, there exists a **discount factor** $\underline{\delta} < 1$ such that, for all $\delta \in (\underline{\delta}, 1)$, there exists a subgame-perfect Nash equilibrium of the infinitely repeated version of the stage game $\mathcal{G}$ that achieves $\{\hat{u}_k\}_{k=1}^K$ as the average payoffs.



# Repeated games: An example [34]

Let's suppose to play the continuous-power near-far power control game an **infinite** number of times:




There exists a critical $\underline{\delta}$ such that, for all $\delta > \underline{\delta}$ (i.e., for **delay-tolerant** users), the grim-trigger strategy achieves the maximum social welfare

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