UNIVERSITÀ DI PISA

Electromagnetic Radiations and Biological Interactions

“Laurea Magistrale” in Biomedical Engineering
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Math background

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Microwave&Radiation Laboratory

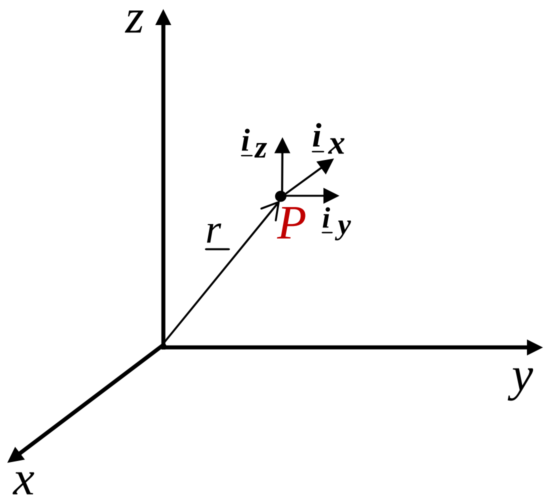
Lecture Content

- **Coordinate systems**
  - Cartesian coordinate system
  - Cylindrical coordinate system
  - Spherical coordinate system
  - Correspondence between coordinate systems
- **Vectors and Vectors Algebra**
- **Boundary conditions**

Cartesian coordinate system

Is a system to uniquely determine the position of a point in space with respect to a reference point called "origin". Each point P is defined by the intersection between three perpendicular surfaces; the versors indicate the direction of a directed axis. Each versor is perpendicular to one of the above surfaces and passes through the point P.

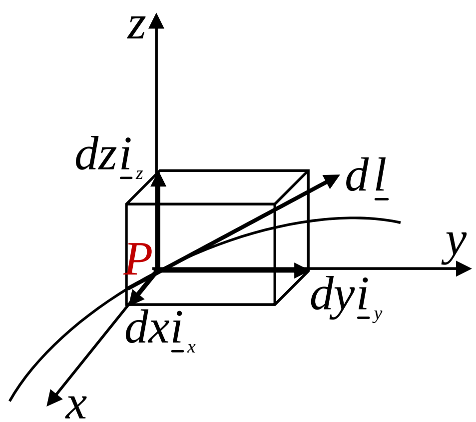
Cartesian coordinate system

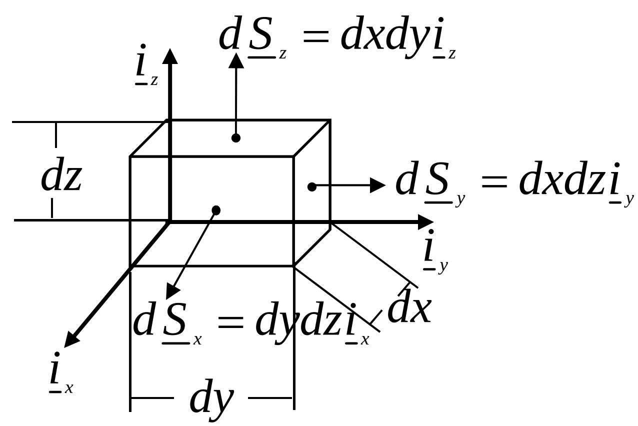


$$P \equiv (x, y, z)$$

$\underline{i}_x, \underline{i}_y, \underline{i}_z$
- determine the versors direction
- do not change their directions with changes of point P in space

$\underline{r} = x\underline{i}_x + y\underline{i}_y + z\underline{i}_z$ - position vector



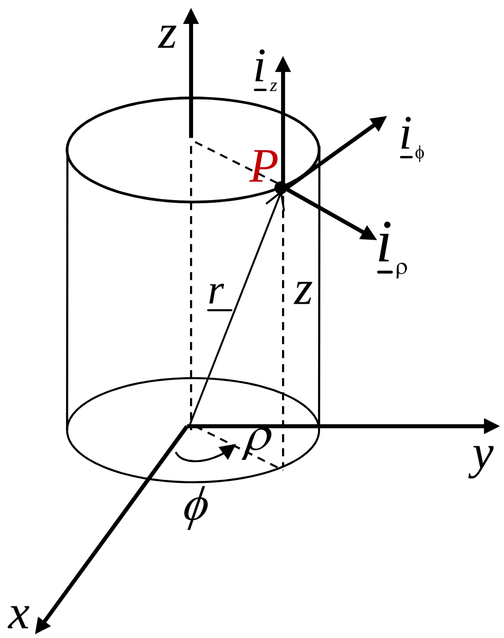


Differential elements:

$$\begin{cases} d\underline{S}_x = dydz\underline{i}_x \\ d\underline{S}_y = dxdz\underline{i}_y \\ d\underline{S}_z = dxdy\underline{i}_z \\ dV = dxdydz \\ d\underline{l} = \underline{i}_x dx + \underline{i}_y dy + \underline{i}_z dz \end{cases}$$

Cylindrical coordinate system

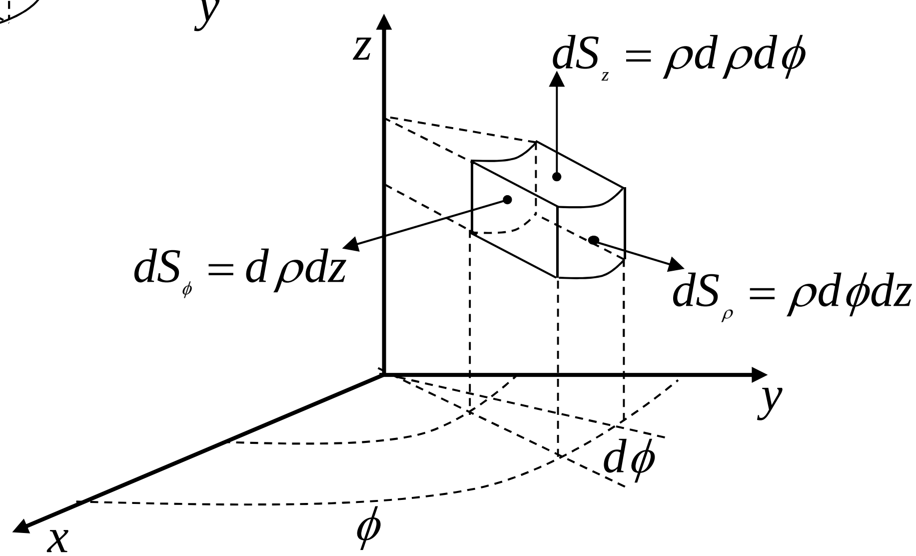
<u>Cylindrical coordinate system</u> *(it is useful when the studied system is symmetric with respect to z axis)*



$$P \equiv (\rho, \phi, z)$$

$\underline{i}_\rho, \underline{i}_\phi, \underline{i}_z$
- determine the versor directions
- $\underline{i}_\rho, \underline{i}_\phi$ change their directions from point to point

$\underline{r} = \rho \underline{i}_\rho + z \underline{i}_z$ - position vector

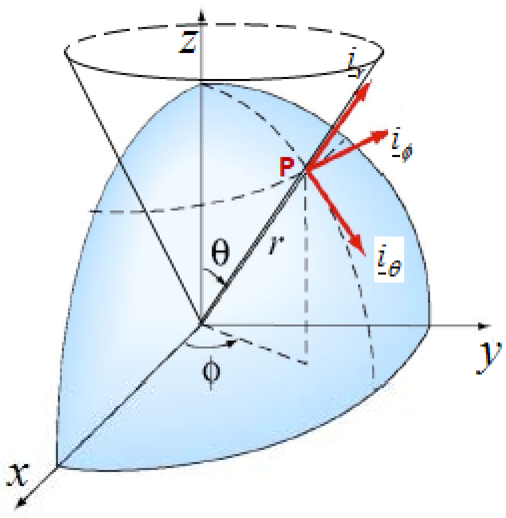


<u>Differential elements:</u>

$$\begin{cases} d\underline{S}_\rho = \rho d\phi dz \underline{i}_\rho \\ d\underline{S}_\phi = d\rho dz \underline{i}_\phi \\ d\underline{S}_z = \rho d\rho d\phi \underline{i}_z \\ dV = \rho d\rho d\phi dz \\ d\underline{l} = \underline{i}_\rho d\rho + \rho \underline{i}_\phi d\phi + \underline{i}_z dz \end{cases}$$

Spherical coordinate system

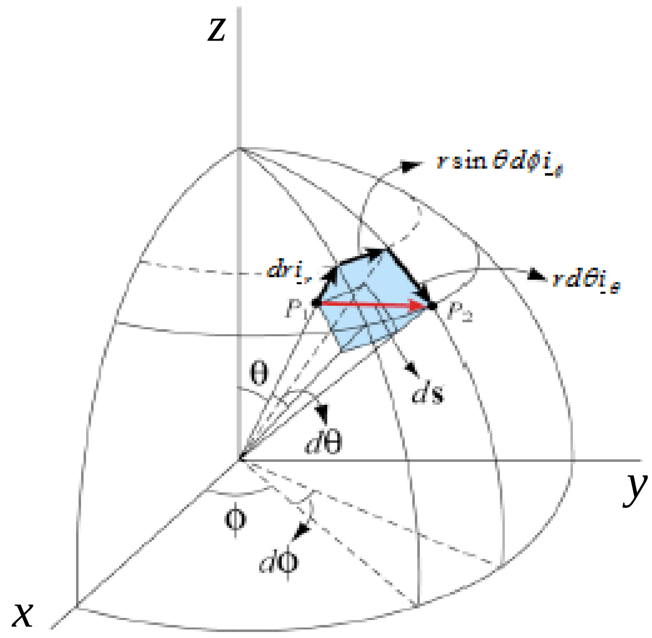
Spherical coordinate system *(it is useful when the studied system is symmetric with respect to origin)*



$P \equiv (r, \theta, \phi)$
- r denotes the distance from the origin
- (θ, ϕ) indicate the direction of the P point location

$\underline{i}_r, \underline{i}_\theta, \underline{i}_\phi$
- determine the versors direction
- change their directions from point to point

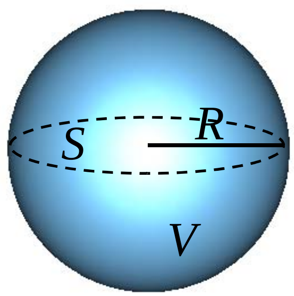
$\underline{r} = r\underline{i}_r$ - position vector



Differential elements:

$$d\underline{S}_r = r^2 d\phi dr \underline{i}_r$$
$$d\underline{S}_\theta = r\sin\theta dr d\phi \underline{i}_\theta$$
$$d\underline{S}_\phi = r d\theta dr \underline{i}_\phi$$
$$dV = r^2 \sin\theta dr d\theta d\phi$$
$$d\underline{l} = \underline{i}_r dr + r\underline{i}_\theta d\theta + r\sin\theta \underline{i}_\phi d\phi$$

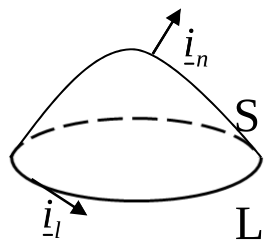
Examples



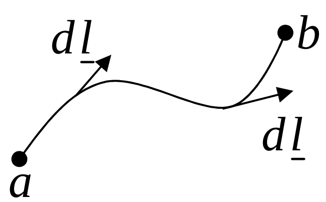
1. Sphere surface area: $S = \iint_S dS = \int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta d\theta d\phi = 4\pi R^2$

2. Sphere volume: $V = \iiint_V dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2 \sin\theta dr d\theta d\phi = \frac{4}{3}\pi R^3$

3. Weight: $M = \iiint_V \rho(P) dV = \int_0^{R}\int_0^{2\pi}\int_0^{\pi} \rho(r,\theta,\phi) r^2 \sin\theta d\theta d\phi$

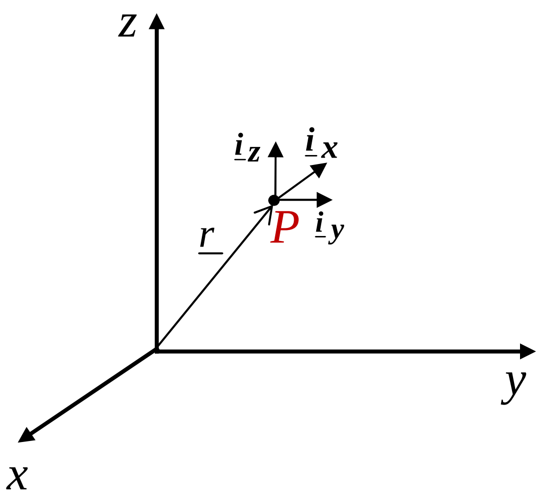


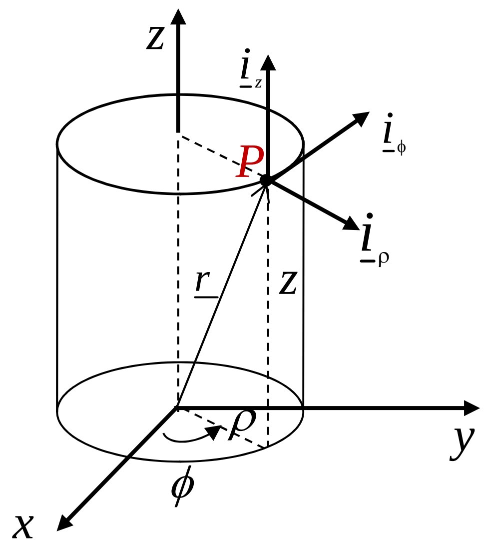
4. Gauss law: $\oint_S \varepsilon_0 \underline{e}\cdot\underline{i}_n dS = Q = \iiint_V \rho dV$

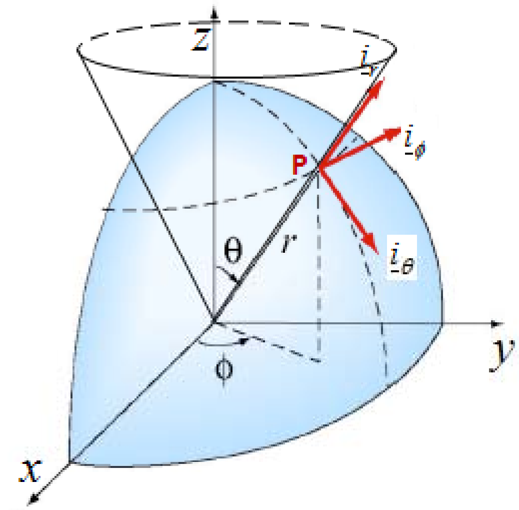


5. Electrostatic field: $\int_a^b \underline{e}\cdot d\underline{l} = \Phi_b - \Phi_a$

Correspondence between coordinate systems (I)

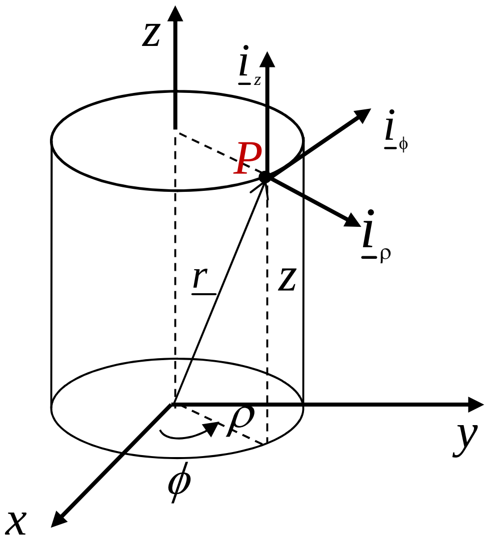


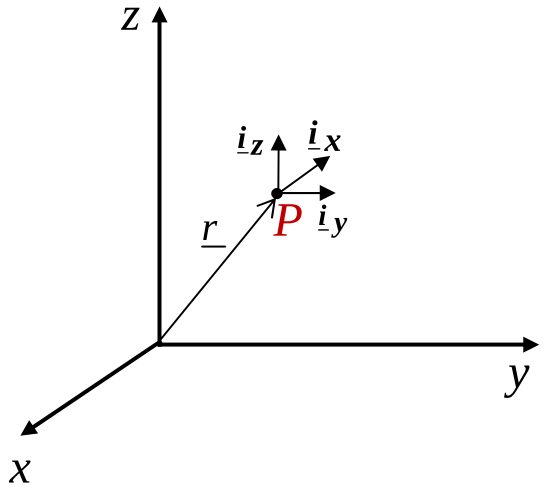


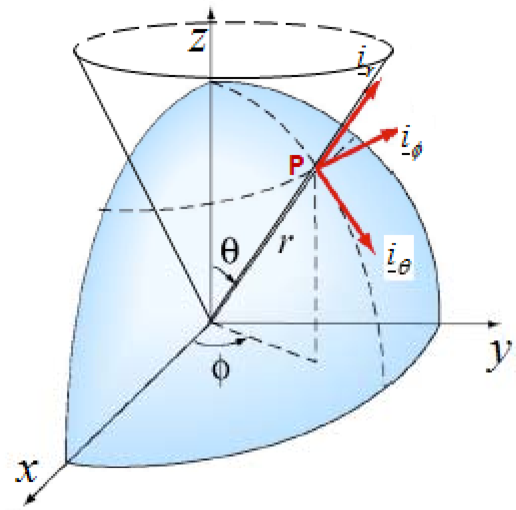


Cartesian	Cylindrical	Spherical
x	$\rho\cos\phi$	$r\sin\theta\cos\phi$
y	$\rho\sin\phi$	$r\sin\theta\sin\phi$
z	z	$r\cos\theta$
$\underline{i}_x$	$\cos\phi\underline{i}_\rho - \sin\phi\underline{i}_\phi$	$\sin\theta\cos\phi\underline{i}_r + \cos\theta\cos\phi\underline{i}_\theta - \sin\phi\underline{i}_\phi$
$\underline{i}_y$	$\sin\phi\underline{i}_\rho + \cos\phi\underline{i}_\phi$	$\sin\theta\sin\phi\underline{i}_r + \cos\theta\sin\phi\underline{i}_\theta + \cos\phi\underline{i}_\phi$
$\underline{i}_z$	$\underline{i}_z$	$\cos\theta\underline{i}_r - \sin\theta\underline{i}_\theta$

Correspondence between coordinate systems (II)

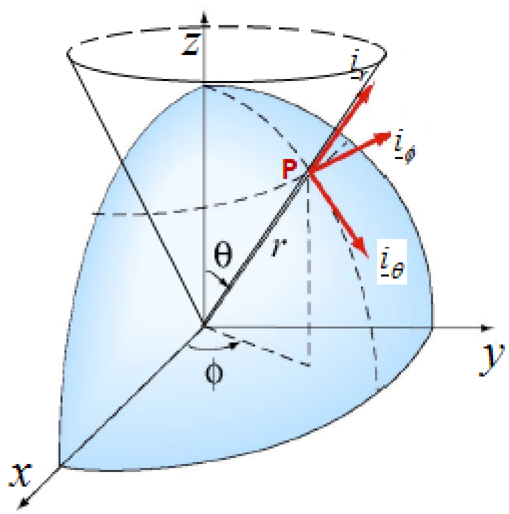


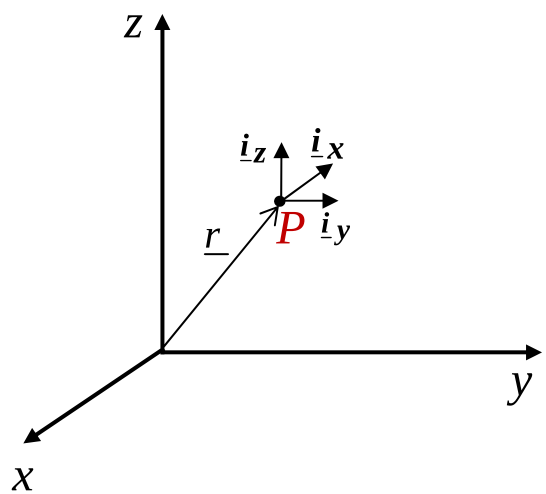


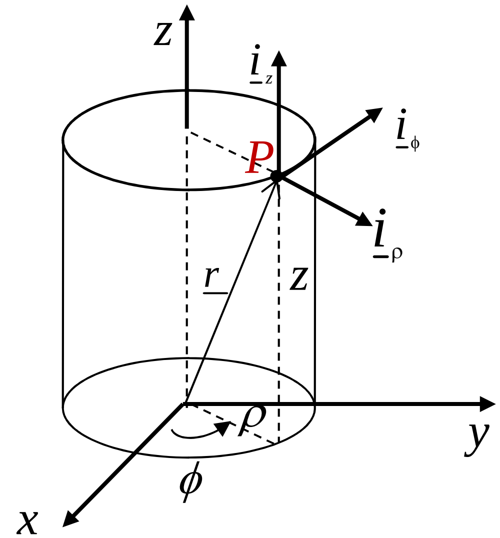


Cylindrical	Cartesian	Spherical
ρ	$\sqrt{x^2 + y^2}$	$r \sin\theta$
ϕ	$\arctan(y/x)$	ϕ
z	z	$r\cos\theta$
$\underline{i}_\rho$	$\cos\phi\,\underline{i}_\rho - \sin\phi\,\underline{i}_\phi$	$\sin\theta\,\underline{i}_r + \cos\theta\,\underline{i}_\theta$
$\underline{i}_\phi$	$-\sin\phi\,\underline{i}_x + \cos\phi\,\underline{i}_y$	$\underline{i}_\phi$
$\underline{i}_z$	$\underline{i}_z$	$\cos\theta\,\underline{i}_r - \sin\theta\,\underline{i}_\theta$

Correspondence between coordinate systems (III)



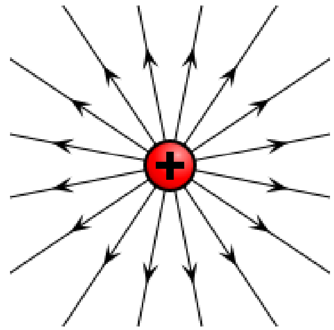




Spherical	Cartesian	Cylindrical
r	$\sqrt{x^2+y^2+z^2}$	$\sqrt{\rho^2+z^2}$
θ	$\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$	$\cos^{-1}\left(\frac{z}{\sqrt{\rho^2+z^2}}\right)$
ϕ	$tg^{-1}(x/y)$	ϕ
$\underline{i}_r$	$\sin\theta\cos\phi\underline{i}_x+\sin\theta\sin\phi\underline{i}_y+\cos\theta\underline{i}_z$	$\sin\theta\underline{i}_\rho+\cos\theta\underline{i}_\theta$
$\underline{i}_\theta$	$\cos\theta\cos\phi\underline{i}_x+\cos\theta\sin\phi\underline{i}_y-\sin\theta\underline{i}_z$	$\cos\theta\underline{i}_\rho-\sin\theta\underline{i}_z$
$\underline{i}_\phi$	$-\sin\phi\underline{i}_x+\cos\phi\underline{i}_y$	$\underline{i}_\phi$

Coordinate system correspondence - example

Electric field intensity

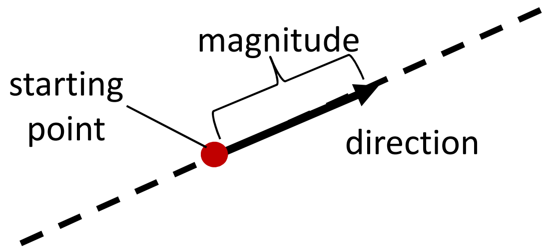


$$\underline{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\left(x\underline{i}_x + y\underline{i}_y + z\underline{i}_z\right)}{(x^2+y^2+z^2)^{3/2}} = \underbrace{\frac{Q}{4\pi\varepsilon_0}\frac{\left(x\underline{i}_x + y\underline{i}_y + z\underline{i}_z\right)}{\sqrt{x^2+y^2+z^2}}\frac{1}{(x^2+y^2+z^2)}}_{\text{Cartesian coordinates}}$$

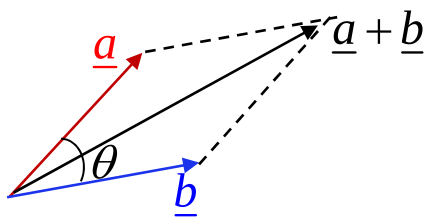
Since charge has spherical symmetry: $$\underline{E} = \underbrace{\frac{Q}{4\pi\varepsilon_0 r^2}\underline{r}}_{\text{Spherical coordinates}}$$

Vectors and Vectors Algebra (I)

Vector: is a geometric element represented by an oriented segment with an arrow at one end. Has both length (magnitude) and direction.

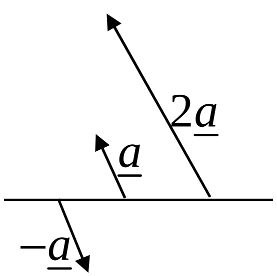


Vectors addition:



$$\left|\underline{a}+\underline{b}\right| = \sqrt{a^2 + b^2 - 2ab\cos\theta}$$

Multiplication by a Scalar:



$\underline{a}$ -vector

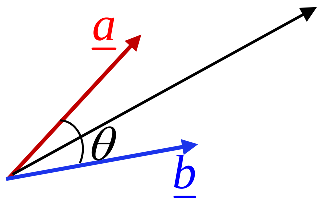
k -scalar

If: $k > 0 \Longrightarrow k\underline{a}$

$k < 0 \Longrightarrow -k\underline{a}$

Vectors and Vectors Algebra (II)

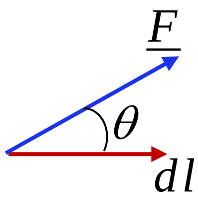
Scalar product: $\underline{a} \cdot \underline{b} = ab\cos\theta$



Properties:

$$\begin{cases} \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \\ \underline{a} \cdot \underline{b} = 0 \ \text{ if: } \underline{a} \perp \underline{b} \end{cases}$$

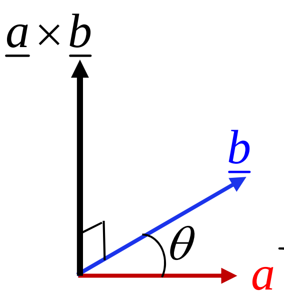
EX: One important physical application of the scalar product is the calculation of work:



$$dW = \underline{F} \cdot d\underline{l} = Fd\cos\theta$$

Work done by constant force, straight line motion

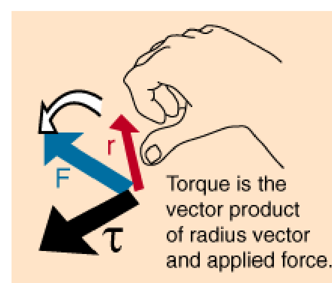
Vector product: $|\underline{a} \times \underline{b}| = ab\sin\theta$



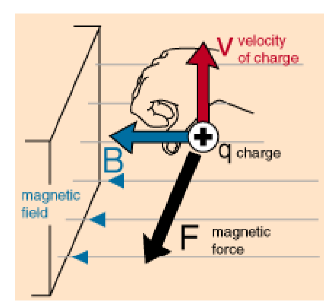
Properties:

$$\begin{cases} (\underline{a}+\underline{b})\times\underline{c} = \underline{a}\times\underline{c} + \underline{b}\times\underline{c} \\ \underline{a}\times\underline{b} = -\underline{a}\times\underline{b} \\ \underline{a}\times\underline{b} = 0 \ \text{ if: } \underline{a} \parallel \underline{b} \\ \underline{a}\times(\lambda\underline{b}) = \lambda(\underline{a}\times\underline{b}) = (\lambda u)\times v \\ \underline{a}\times(\underline{b}\times\underline{c}) = b(a\cdot c) - c(a\cdot b) \\ (\underline{a}\times\underline{b})\cdot(\underline{a}\times\underline{c}) = a^2(b\cdot c) - (a\cdot b)(a\cdot c) \end{cases}$$

EX: vector product appears in the calculation of torque and in the calculation of the magnetic force on a moving charge.

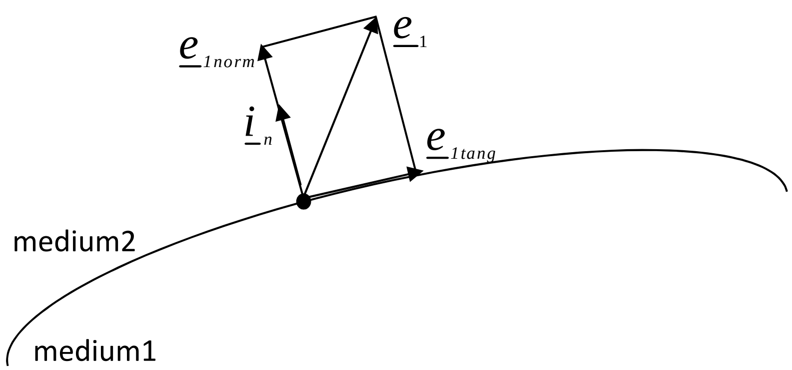


$$\underline{\tau} = \underline{r} \times \underline{F} = rF\sin\theta$$



$$\underline{F} = q\underline{v} \times \underline{B} = qvB\sin\theta$$

Boundary conditions



<u>*Boundary conditions of the tangent electric*</u>

$$\left.\begin{array}{l}\left(\underline{i}_n \times \underline{e}_2\right) = \left(\underline{i}_n \times \underline{e}_1\right) \\ \underline{e}_1 = \underline{e}_{1norm} + \underline{e}_{1tang} \\ \underline{e}_2 = \underline{e}_{2norm} + \underline{e}_{2tang}\end{array}\right| \Longrightarrow \underline{e}_{2tang} = \underline{e}_{1tang}$$

<u>*Boundary conditions of the normal component of electric induction*</u>

$$\left.\begin{array}{l}\underline{i}_n \cdot \underline{d}_1 = \underline{i}_n \cdot \underline{d}_2 \\ \underline{d}_1 = \varepsilon_1 \cdot \underline{e}_1 \\ \underline{d}_2 = \varepsilon_2 \cdot \underline{e}_2\end{array}\right| \Longrightarrow \underline{i}_n \cdot \varepsilon_1 \underline{e}_1 = \underline{i}_n \cdot \varepsilon_2 \underline{e}_2$$

References

1. G. Manara, A. Monorchio, P.Nepa, “*Appunti di Campi Elettromagnetici*”
2. J. Slater, N. Frank, “*Electromagnetism*”
3. M. Schwartz, “*Principle of Electrodynamics*”
4. http://www.wikipedia.org/