UNIVERSITÀ DI PISA

Electromagnetic Radiations and Biological Interactions

"Laurea Magistrale" in Biomedical Engineering
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Fields and Differential Operators

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Lecture Content

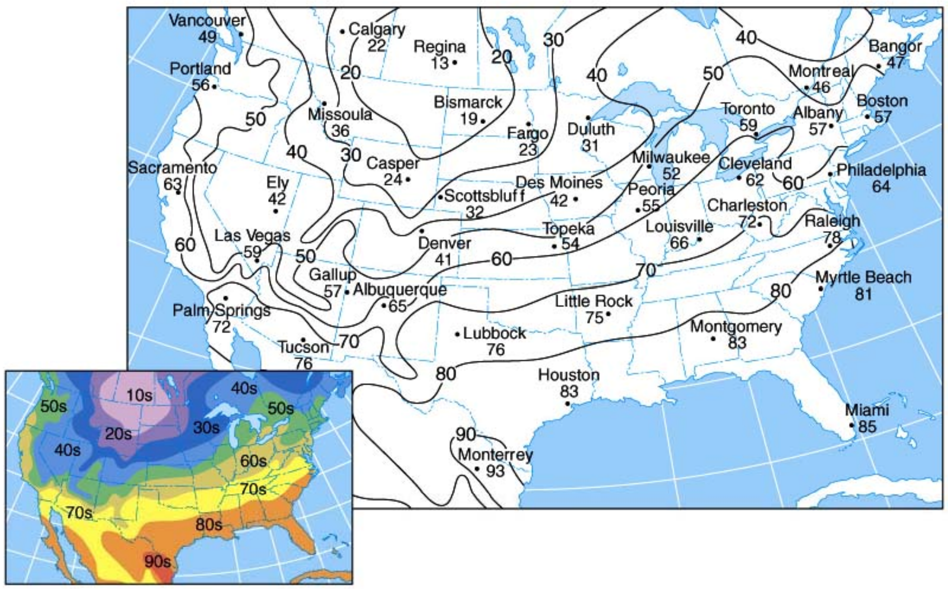
- **Scalar and vector fields**
- **Static and dynamic fields**
- **Frequency and time domain**
- **Phasors**

Scalar fields

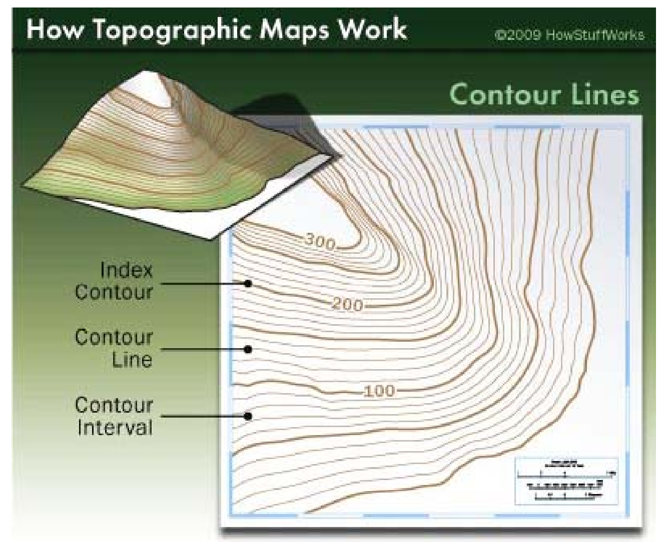
A **scalar field U** is defined in a region of space where it can be defined a scalar function:

$$U = U(P) = U(r)$$

Examples include the temperature distribution throughout space, the pressure distribution in a fluid or in the atmosphere. A scalar field is represented by surfaces (or lines), where U(P)=const.



Temperature map



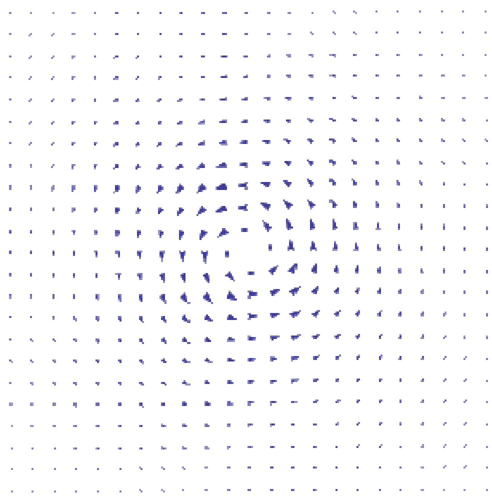
Topographical map

Vector fields

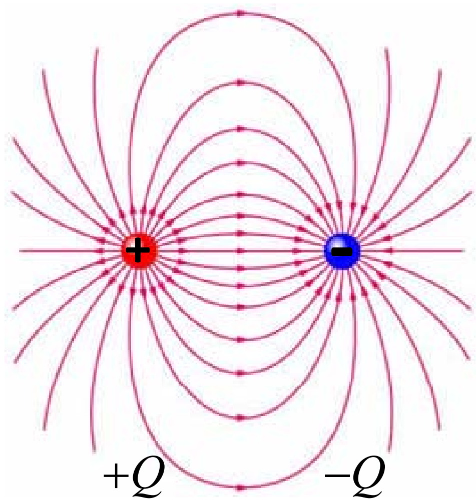
A **vector field U** is defined in a region of space where it can be defined a vector function:

$$\underline{U} = \underline{U}(P) = \underline{U}(\underline{r}) = U_x(\underline{r})\underline{i}_x + U_y(\underline{r})\underline{i}_y + U_z(\underline{r})\underline{i}_z$$

Vector fields are often used to model, for example, the strength and direction of some force, such as the magnetic or gravitational force, the speed and direction of a moving fluid throughout space. This physical intuition leads to notions such as the divergence (which represents the rate of change of volume of a flow) and curl (which represents the rotation of a flow).



An irrotational vector field



Field lines of two point charges

A vector field is described by a set of direction lines, also known as stream lines or flux lines. The direction line is a curve constructed so that the field is tangential to the curve in all points of the curve.

Static and dynamic fields

- The temperature in a room is a field of temperature, composed of the values of temperature in a number of points of the room. We do not see the field, but it exists, and we can for instance visualize constant-temperature or isothermal surfaces.
- If the physical quantity associated with a scalar or vector field is:
  - time independent we are talking about **static fields**.
  - time dependent we are talking about **dynamic fields**. Considering, for instance, the temperature field, the room may be heated or cooled, which makes the temperature field time dependent.

Generally, fields depend on time: $U(\underline{r},t)$ *or* $\underline{U}(\underline{r},t)$

Gradient of a scalar field

The **gradient** of a scalar field is a vector field that points in the direction of the greatest rate of increase of the scalar field, and whose magnitude is the rate of change.

For a scalar field $f(x,y,z)$, the variation df for $(x+dx,\ y+dy,\ z+dz)$:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$d\underline{l} = dx\underline{i}_x + dy\underline{i}_y + dz\underline{i}_z$$

$$\Longrightarrow df = \underbrace{\left(\frac{\partial f}{\partial x}\underline{i}_x + \frac{\partial f}{\partial y}\underline{i}_y + \frac{\partial f}{\partial z}\underline{i}_z\right)}_{\text{gradient of } f:} \cdot d\underline{l}$$

$$grad f = \nabla f = \left(\frac{\partial f}{\partial x}\underline{i}_x + \frac{\partial f}{\partial y}\underline{i}_y + \frac{\partial f}{\partial z}\underline{i}_z\right) \Longrightarrow \nabla = \left(\frac{\partial}{\partial x}\underline{i}_x + \frac{\partial}{\partial y}\underline{i}_y + \frac{\partial}{\partial z}\underline{i}_z\right)$$

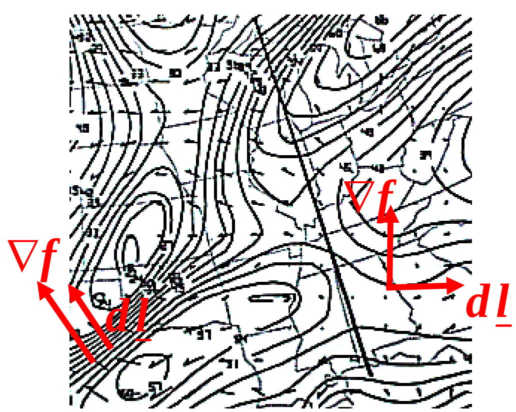
For a length element dl, the max(df) is obtained when $d\underline{l}$ is oriented in the same direction of the gradient:

$$df = \nabla f \cdot d\underline{l} = \left|\nabla f\right| dl \cos\alpha \rightarrow \alpha = \angle\left(\nabla f, d\underline{l}\right) \quad \alpha = 0 \Longrightarrow df = \max(df)$$

(The gradient has the direction of the maximum f variation.)

$$\alpha = \pi/2 \Longrightarrow df = 0$$

(The gradient is perpendicular to surfaces with f=const. and is directed toward increasing level curves)



Gradient - examples

Examples:

1. Consider a room in which the temperature is given by a scalar field, T, so at each point (x,y,z) the temperature is $T(x,y,z)$. (We will assume that the temperature does not change over time.) At each point in the room, the gradient of T at that point will show the direction the temperature rises most quickly. The magnitude of the gradient will determine how fast the temperature rises in that direction.

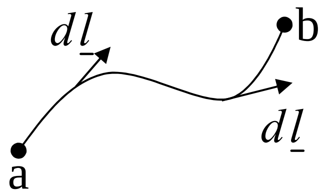
2. Consider a surface whose height above sea level at a point (x,y) is $H(x,y)$. The gradient of H at a point is a vector pointing in the direction of the steepest slope or grade at that point. The steepness of the slope at that point is given by the magnitude of the gradient vector.

Gradient of a scalar field – properties

1. Linearity $\nabla\left(c_1 u + c_2 v\right) = c_1 \nabla u + c_2 \nabla v$

2. Product rule $\nabla\left(uv\right) = u\nabla v + v\nabla u$

3. Division rule $\nabla\left(u/v\right) = \dfrac{v\nabla u - u\nabla v}{v^2}$



$\int_a^b \nabla f \cdot d\underline{l} = \int_a^b df = f(b) - f(a)$ - It doesn't depends on the path from a to b.

Reference systems - correspondence

Gradient in Cartesian coordinate system:

$$\nabla f = \left(\frac{\partial f}{\partial x} \underline{i}_x + \frac{\partial f}{\partial y} \underline{i}_y + \frac{\partial f}{\partial z} \underline{i}_z \right)$$

Gradient in Cylindrical coordinate system:

$$\nabla f = \left(\frac{\partial f}{\partial \rho} \underline{i}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \underline{i}_\phi + \frac{\partial f}{\partial z} \underline{i}_z \right)$$

Gradient in Spherical coordinate system:

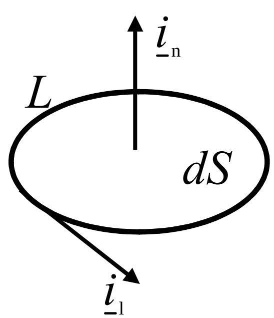
$$\nabla f = \left(\frac{\partial f}{\partial r} \underline{i}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \underline{i}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \underline{i}_\phi \right)$$

Curl of a vector field

The **curl** (or **rotor**) is a vector operator that describes the infinitesimal rotation of a vector field. At every point in the field, the curl is represented by a vector. The attributes of this vector (length and direction) characterize the rotation at that point. The direction of the curl is the axis of rotation, as determined by the right-hand rule, and the magnitude of the curl is the magnitude of rotation.

For a vector field $\underline{A}(P) = A_x \underline{i}_x + A_y \underline{i}_y + A_z \underline{i}_z$ $\Box$ $rot\,\underline{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\underline{i}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\underline{i}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\underline{i}_z$

Considering the "nabla" operator: $rot\,\underline{A} = \nabla \times \underline{A} = \begin{vmatrix} \underline{i}_x & \underline{i}_y & \underline{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$



$$\underline{i}_n \cdot \nabla \times \underline{A} \;\Box\; \lim_{dS \to 0} \frac{\oint_L \underline{A} \cdot d\underline{l}}{dS}$$

Curl of a vector field - properties

$$(1)\ \nabla \times (\nabla f) = 0$$

$$(2)\ \nabla \times a = 0 \Longrightarrow \underline{a} = \nabla u$$ ($\underline{a}$ - irrotational field)

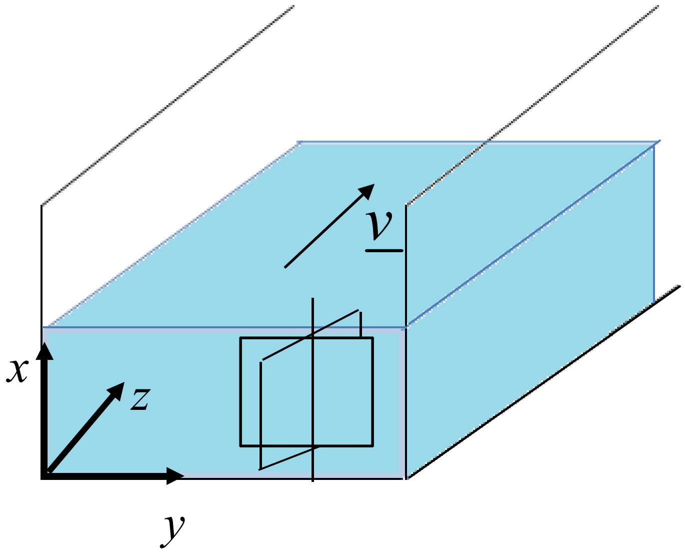
$$(3)\ \nabla \times (\underline{a} + \underline{b}) = \nabla \times \underline{a} + \nabla \times \underline{b}$$

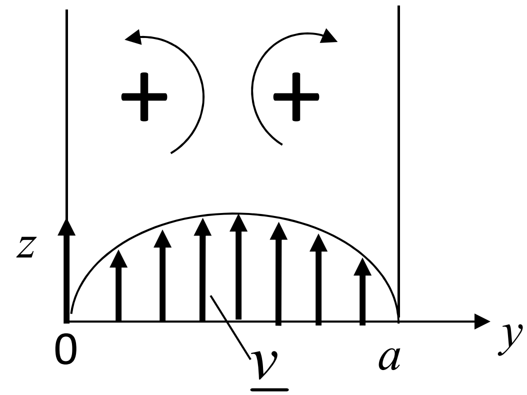
$$(4)\ \nabla \times (f\,\underline{a}) = f\,(\nabla \times \underline{a}) + (\nabla f) \times \underline{a}$$

(f- scalar field; $\underline{a}$ - vector field)

Curl - example

Example: consider a liquid that flows into a pipe .





Consider a wheel inside the liquid, which is small enough to neglect its perturbation to the motion of the liquid. Along the symmetry of the pipe axis the wheel doesn't move (curl is zero). The wheel rotation velocity depends on the x-component of the curl where the wheel is inserted (curl sign indicates the rotation – clockwise/counterclockwise).

Reference systems - correspondence

Curl in Cartesian coordinate system:

$$\nabla\times\underline{A}=\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\underline{i}_x+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\underline{i}_y+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\underline{i}_z$$

Curl in Cylindrical coordinate system:

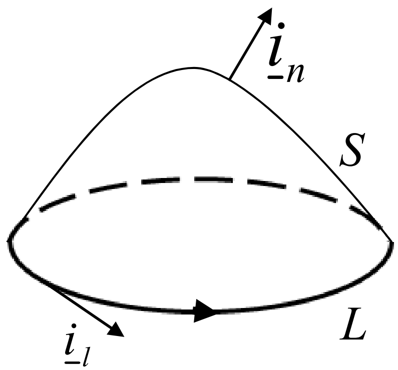
$$\nabla\times\underline{A}=\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right)\underline{i}_\rho+\left(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\right)\underline{i}_\phi+\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}\left(\rho A_\phi\right)-\frac{\partial A_\rho}{\partial\phi}\right)\underline{i}_z$$

Curl in Spherical coordinate system:

$$\nabla\times\underline{A}=\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(A_\phi\sin\theta\right)-\frac{\partial A_\theta}{\partial\phi}\right)\underline{i}_r+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial}{\partial r}\left(rA_\phi\right)\right)\underline{i}_\theta+\frac{1}{r}\left(\frac{\partial}{\partial r}\left(rA_\theta\right)-\frac{\partial A_r}{\partial\theta}\right)\underline{i}_\phi$$

Curl theorem

Stokes's theorem (curl theorem)

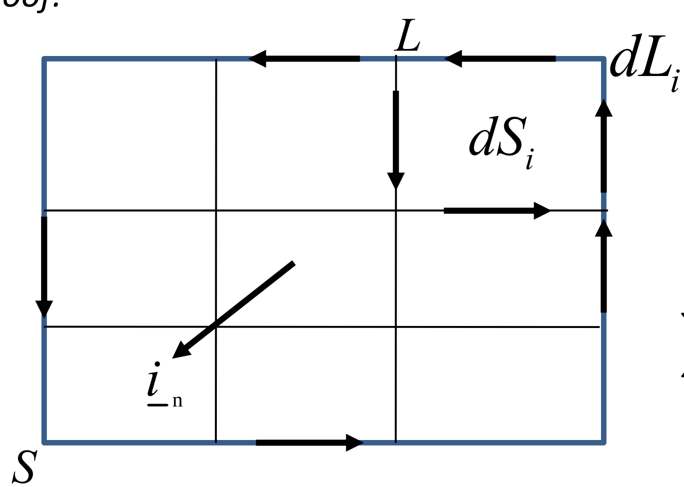


$$\oint_L \underline{A}\cdot d\underline{l} = \iint_S \left(\nabla\times\underline{A}\right)\cdot\underline{i}_n dS$$

Stokes's theorem
($\forall$ surface S whose boundary coincides with L)

($\underline{i}_n$ and L path are related by the right-hand rule)

Proof:



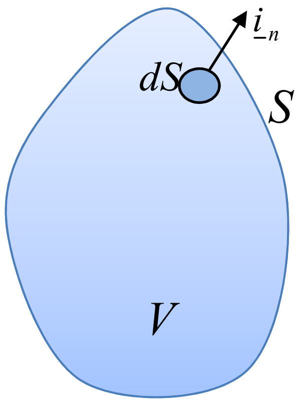
$$\oint_{dL_i} \underline{A}\cdot d\underline{l} = \left(\nabla\times\underline{A}\right)\cdot\underline{i}_n dS_i$$

$$\sum_{i=1}^{N}\oint_{dL_i} \underline{A}\cdot d\underline{l} = \sum_{i=1}^{N}\left(\nabla\times\underline{A}\right)\cdot\underline{i}_n dS_i$$

$\Longrightarrow$ ***Stokes's theorem***

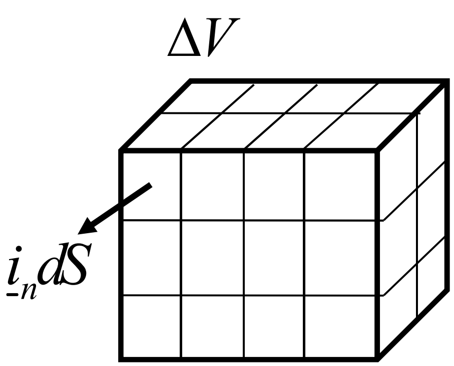
Divergence

The **divergence** is a vector operator that measures the magnitude of a vector field's source or sink at a given point, in terms of a signed scalar. More technically, the divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume around a given point. For example, consider air as it is heated or cooled. The relevant vector field for this example is the velocity of the moving air at a point. If air is heated in a region it will expand in all directions such that the velocity field points outward from that region. Therefore the divergence of the velocity field in that region would have a positive value, as the region is a source. If the air cools and contracts, the divergence is negative and the region is called a sink.



$$\nabla \cdot \underline{A} \triangleq \lim_{\Delta V \to 0} \frac{\oint\!\!\oint_S \underline{A} \cdot \underline{i}_n dS}{\Delta V}$$ ***Divergence***

Proof:



For a vector field $\underline{A}(P) = A_x \underline{i}_x + A_y \underline{i}_y + A_z \underline{i}_z$ $\quad div \underline{A} \triangleq \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

Considering the "nabla" operator: $div \underline{A} = \nabla \cdot \underline{A} = \left(\frac{\partial}{\partial x} \underline{i}_x + \frac{\partial}{\partial y} \underline{i}_y + \frac{\partial}{\partial z} \underline{i}_z \right) \cdot \underline{A}$

Divergence - properties

$$(1)\nabla\cdot\left(\nabla\times\underline{a}\right)=0$$

$$(2)\nabla\cdot\underline{b}=0 \Longrightarrow \underline{b}=\nabla\times\underline{a}$$ ($\underline{b}$- soleinodal field)

$$(3)\nabla\cdot\left(\underline{a}\times\underline{b}\right)=\underline{b}\cdot\left(\nabla\times\underline{a}\right)-\underline{a}\cdot\left(\nabla\times\underline{b}\right)$$

$$(4)\nabla\cdot\left(u\underline{a}\right)=\left(\nabla u\right)\cdot\underline{a}+u\left(\nabla\cdot\underline{a}\right)$$

$$(5)\nabla\cdot\left(\underline{a}+\underline{b}\right)=\nabla\cdot\underline{a}+\nabla\cdot\underline{b}$$

<u>Helmholtz's theorem:</u> $\underline{a}=\underline{a}_i+\underline{a}_s$ with $\nabla\times\underline{a}_i=0$

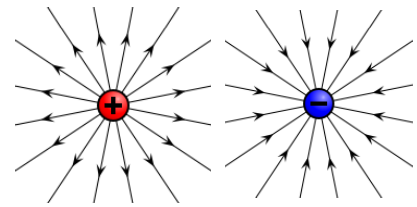
$\nabla\cdot\underline{a}_i=0$

$\underline{a}_i$: irrotational field

$\underline{a}_s$: solenoidal field

Divergence -example

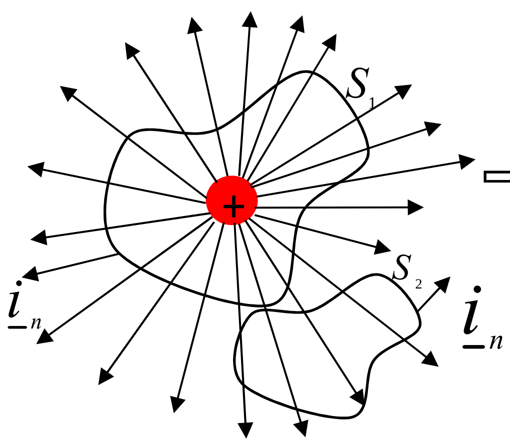
Example: the electrical charge generates an electric field:



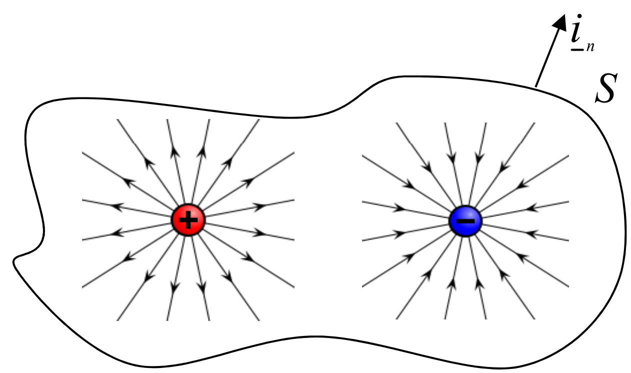
If at a point P there is no charge:

$$\nabla \cdot \underline{E} = 0$$

If: $\oiint_S \underline{E} \cdot \underline{i}_n dS \neq 0 \Rightarrow$ $(\exists)$ charge inside of the volume V



$$\Rightarrow \begin{array}{l} \iint_{S_1} \underline{E} \cdot \underline{i}_n dS = Q \\ \iint_{S_2} \underline{E} \cdot \underline{i}_n dS = 0 \end{array}$$



$$\Rightarrow \iint_S \underline{E} \cdot \underline{i}_n dS = 0$$

Reference systems - correspondence

Divergence in Cartesian coordinate system:

$$\nabla \cdot \underline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Divergence in Cylindrical coordinate system:

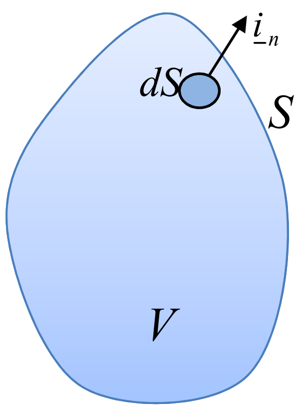
$$\nabla \cdot \underline{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho A_\rho\right) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Divergence in Spherical coordinate system:

$$\nabla \cdot \underline{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Divergence theorem

Gauss's theorem (divergence theorem)

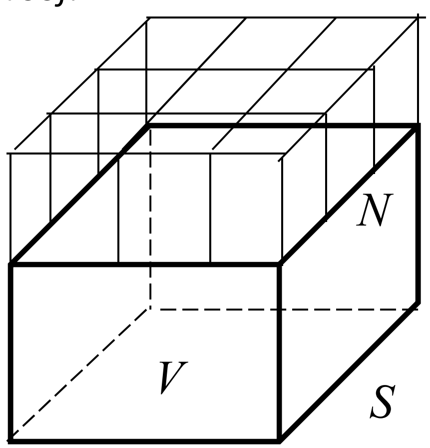


$$\oint\!\!\!\oint_S \underline{A}\cdot\underline{i}_n dS = \iiint_V \left(\nabla\cdot\underline{A}\right) dV$$

Gauss's theorem

$\underline{i}_n$-is pointing in the direction outward the volume

Proof:



$$\oint\!\!\!\oint_{\Delta S_i} \underline{A}\cdot\underline{i}_n dS = \left(\nabla\cdot\underline{A}\right)\Delta V_i$$

$$\sum_{i=1}^{N}\oint\!\!\!\oint_{\Delta S_i} \underline{A}\cdot\underline{i}_n dS = \sum_{i=1}^{N}\left(\nabla\cdot\underline{A}\right)\Delta V_i$$

$\Longrightarrow$ ***Gauss's theorem***

Laplace operator

The **Laplacian** operator is a differential operator given by the divergence of the gradient of a function on Euclidean space.

For an f scalar function: $\nabla \cdot (\nabla f) = \nabla^2 f$

Laplacian

In Cartesian coordinates: $\nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

If $\nabla^2 f = 0 \longrightarrow f$ - harmonic function

Vector Laplacian: $\nabla^2 \underline{a} = \nabla \cdot (\nabla \cdot \underline{a}) - \nabla \times \nabla \times \underline{a}$

In Cartesian coordinates: $\nabla^2 \underline{a} = (\nabla^2 \cdot a_x)\underline{i}_x + (\nabla^2 \cdot a_y)\underline{i}_y + (\nabla^2 \cdot a_z)\underline{i}_z$

Laplace operator - examples

1. Laplace's and Poisson's Equations

- A useful approach to the calculation of electric potentials is to relate that potential to the charge density which gives rise to it. The electric field is related to the charge density by the divergence relationship:

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

E-electric field
ρ-charge density
ε_0-permittivity

and the electric field is related to the electric potential by a gradient relationship: $E = -\nabla \Phi$

- The potential is related to the charge density by Poisson's equation: $\nabla \cdot \nabla \Phi = \nabla^2 \Phi = -\frac{\rho}{\varepsilon_0}$

- In a charge-free region of space, this becomes Laplace's equation: $\nabla^2 \Phi = 0$

2. Wave Equation $\nabla^2 \phi + \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0$

Reference systems - correspondence

Laplacian in Cartesian coordinate system:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x} + \frac{\partial^2 f}{\partial y} + \frac{\partial^2 f}{\partial z}$$

Laplacian in Cylindrical coordinate system:

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Laplacian in Spherical coordinate system:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

Frequency & time domain

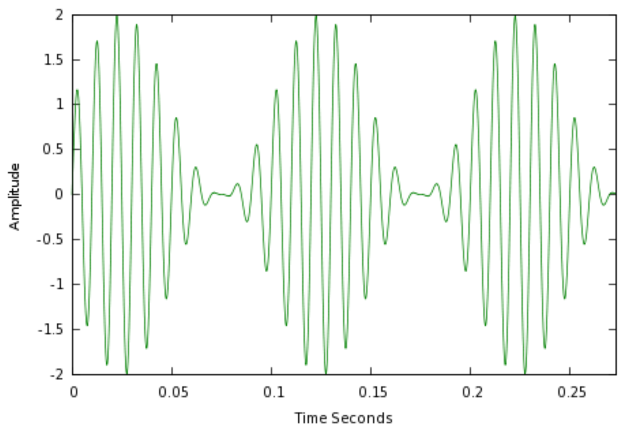
- **<u>Frequency-domain</u>** descriptions are extremely useful when investigating monochromatic or narrow-band phenomena, when the materials and systems of interest are submitted to a source of sinusoidal fields. The quantities they evaluate are complex numbers or vectors. They are not real, physically measurable quantities (because a sinusoidal source is not physical), although they may have a real physical content. the general description in the frequency domain implies complex quantities, with a real and an imaginary part, respectively, which are not physical either.

- **<u>Time-domain</u>** descriptions evaluate the effect of physical sources, where the phenomena are described as a function of time and hence they are real and physically measurable. They must be used when investigating wide-band phenomena.

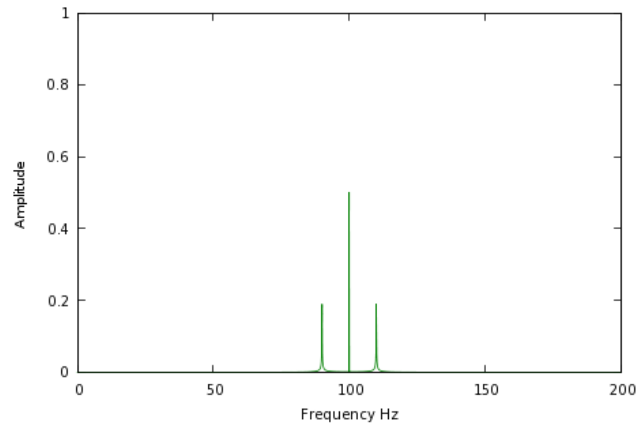
✓The interaction of RF/microwave fields with biological tissues is investigated mostly in the *frequency domain*, with sources considered as sinusoidal.
✓ Today numerical signals, such as for telephony, television, and frequency modulated (FM) radio, may, however, necessitate *time domain* analyses and measurements.

Frequency & time domain - examples

Time domain



Frequency domain



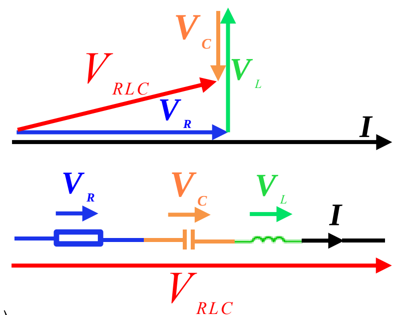
$$x(t) \Leftrightarrow X(f)$$

$$X(f) = F[x(t)]$$ (Fourier Transform)

$$x(t) = F^{-1}[x(f)]$$ (Fourier Inverse Transform)

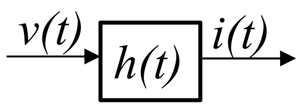
Phasors (I)

Phasor is a representation of a sine wave whose amplitude (A), phase (φ), and angular frequency (ω) are time-invariant. Phasors reduce the dependencies on these parameters to three independent factors, thereby simplifying certain kinds of calculations. The frequency factor, which also includes the time-dependence of the sine wave, is often common to all the components of a linear combination of sine waves. Using phasors, it can be factored out, leaving just the static amplitude and phase information to be combined algebraically (rather than trigonometrically).



An example of series RLC circuit and respective phasor diagram for a specific ω.

A system can be outlined:



If the input to a linear circuit is a sinusoid, then the output from the circuit will be a sinusoid. Specifically, if we have a voltage sinusoid:

$$v(t) = A_M \cos(\omega t + \varphi)$$

Then the current through the linear circuit will also be a sinusoid, although its magnitude and phase may be different quantities: $i(t) = A_N \cos(\omega t + \varphi)$

An impulse response of a system can be used to determine the output of the system: $y(t) = h(t) * x(t)$
- $x(t)$- input
- $h(t)$- impulse response
- $y(t)$- output

In frequency domain: $Y(f) = H(f)X(f)$

$$X(f) = F[x(t)]$$

Fourier Transform

In time domain: $y(t) = F^{-1}[Y(f)]$

Inverse Fourier Transform

Phasors (II)

*Phasors are used for a **sinusoidal signal representation**.*

$$a(t) = A_M \cos(\omega t + \varphi) \Longrightarrow \text{phasor: } A = A_M e^{j\varphi} \Longrightarrow a(t) = \mathrm{Re}\{A_M e^{j\omega t}\} = \mathrm{Re}\{A e^{j(\varphi+\omega t)}\}$$

A phasor associated with a vector field:

$$\underline{e}(t) = E_{x_M}\cos(\omega t + \varphi_x)\hat{x} + E_{y_M}\cos(\omega t + \varphi_y)\hat{y} + E_{z_M}\cos(\omega t + \varphi_z)\hat{z} \Rightarrow$$

$$\underline{E} = E_{x_M} e^{j\varphi_x}\hat{x} + E_{y_M} e^{j\varphi_y}\hat{y} + E_{z_M} e^{j\varphi_z}\hat{z}$$

$$\Updownarrow$$

$$\underline{E} = [E_{x_M}\cos\varphi_x \hat{x} + E_{y_M}\cos\varphi_y \hat{y} + E_{z_M}\cos\varphi_z \hat{z}] + j[E_{x_M}\sin\varphi_x \hat{x} + E_{y_M}\sin\varphi_y \hat{y} + E_{z_M}\sin\varphi_z \hat{z}] = \underline{A} + j\underline{B}$$

In time domain: $\underline{e}(t) = \mathrm{Re}\{\underline{E} e^{j\omega t}\} = \underline{A}\cos(\omega t) - \underline{B}\sin(\omega t)$

Temporal variation:

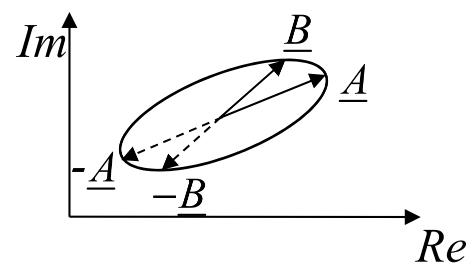
$t = 0 \Rightarrow E = A$

$t = \frac{T}{4} \Rightarrow E = -B$

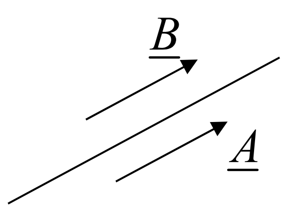
$t = \frac{T}{2} \Rightarrow E = -A$

$t = \frac{3T}{4} \Rightarrow E = B$

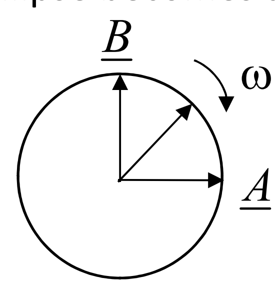
⇨ the $\underline{e}$ vector draws an ellipse:



$\underline{A}$ □ $\underline{B}$ ⇨ the ellipse becomes a line



$\underline{A} \perp \underline{B}$ ⇨ the ellipse becomes a circumference



! A sinusoidal field generates a vector field oscillating along a line, circumference or ellipse.

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