Electromagnetic Radiations and Biological Interactions

"Laurea Magistrale" in Biomedical Engineering
First semester (6 credits, 60 hours), academic year 2011/12

Prof. Paolo Nepa
p.nepa@iet.unipi.it

Maxwell's Equations

Lecture Content

- **Maxwell's equations**
  - -**Time domain**
    - **Differential form**
    - **Integral form**
  - - **Frequency domain**
    - **Differential form**
    - **Integral form**
  - -**Boundary conditions**

Introduction (I)

James Clerk Maxwell

- James Clerk Maxwell (1831–1879) was a Scottish physicist and mathematician.
- His most prominent achievement was formulating **classical electromagnetic theory**. This united all previously unrelated observations, experiments and equations of electricity, magnetism and even optics into a consistent theory.
- Maxwell's equations demonstrated that electricity, magnetism and even light are all manifestations of the same phenomenon, namely the electromagnetic field.
- Subsequently, all other classic laws or equations of these disciplines became simplified cases of Maxwell's equations.
- Maxwell's achievements concerning electromagnetism have been called the "second great unification in physics“, after the first one realized by Isaac Newton.

Introduction (II)

- Maxwell’s equations are a set of four partial differential equations in four variables that fully describe the classical electromagnetic interaction.

- Maxwell's equations describe how electric charges and electric currents act as sources for the electric and magnetic fields. Further, they describe how a time varying electric field generates a time varying magnetic field and vice versa.

- Two of Maxwell's equations -*Gauss's law* and *Gauss's law for magnetism*- describe how the fields emanate from charges. (For the magnetic field there is no magnetic charge and therefore magnetic fields lines neither begin nor end anywhere.)

- The other two Maxwell's equations describe how the fields “circulate” around their respective sources. In *Ampère's law* the magnetic field “circulates” around time varying electric fields. In *Faraday's law* the electric field “circulates” around time varying magnetic fields.

Maxwell's Equations - Differential form (I)

$$\nabla \times \underline{e}(\underline{r},t) = -\frac{\partial}{\partial t}\underline{b}(\underline{r},t)$$

$$\nabla \times \underline{h}(\underline{r},t) = \frac{\partial}{\partial t}\underline{d}(\underline{r},t) + \underline{j}(\underline{r},t)$$

$$\nabla \cdot \underline{d}(\underline{r},t) = \rho(\underline{r},t)$$

$$\nabla \cdot \underline{b}(\underline{r},t) = 0$$

$\underline{e}(\underline{r},t)$ - electric field [V/m]
$\underline{h}(\underline{r},t)$ - magnetic field [A/m]
$\underline{b}(\underline{r},t)$ - magnetic induction field [Tesla/m^2 or Wb/m^2]
$\underline{d}(\underline{r},t)$ - electric induction field [C/m^2]
$\rho(\underline{r},t)$ - charge density [C/m^3]
$\underline{j}(\underline{r},t)$ - current density [A/m^2]

Maxwell's Equations - Differential form (II)

! Maxwell's equations are not independent equations.

$$\nabla \times \underline{e} = -\frac{\partial}{\partial t}\underline{b}(\underline{r},t) \quad \& \quad \nabla \cdot \underline{b}(\underline{r},t) = 0$$

$$\nabla \cdot (\nabla \times \underline{e}) = -\nabla \cdot \left[\frac{\partial}{\partial t}\underline{b}(\underline{r},t) \right]$$

$$\nabla \cdot (\nabla \times \underline{a}) = 0$$

$$\Longrightarrow \nabla \cdot \frac{\partial \underline{b}}{\partial t} = 0 \Longleftrightarrow \frac{\partial}{\partial t}(\nabla \cdot \underline{b}) = 0$$

$\downarrow$ constant in time

$$t < t_0 \Longrightarrow \underline{b}(t < t_0) = 0 \Longrightarrow \nabla \cdot \underline{b} = 0 \quad \nabla t$$

Maxwell's Equations - Differential form (III)

Maxwell's equations (2 and 3) are related through the continuity equation:

$$\nabla \times \underline{h}(\underline{r},t) = \frac{\partial}{\partial t}\underline{d}(\underline{r},t) + \underline{j}(\underline{r},t) \;\; \& \;\; \nabla \cdot \underline{d}(\underline{r},t) = \rho(\underline{r},t)$$

$$\nabla \cdot [\nabla \times \underline{h}(\underline{r},t)] = \nabla \cdot \frac{\partial}{\partial t}\underline{d}(\underline{r},t) + \nabla \cdot \underline{j}(\underline{r},t)$$

$$\nabla \cdot (\nabla \times \underline{a}) = 0$$

$$\Rightarrow \nabla \cdot \frac{\partial}{\partial t}\underline{d}(\underline{r},t) + \nabla \cdot \underline{j}(\underline{r},t) = 0$$

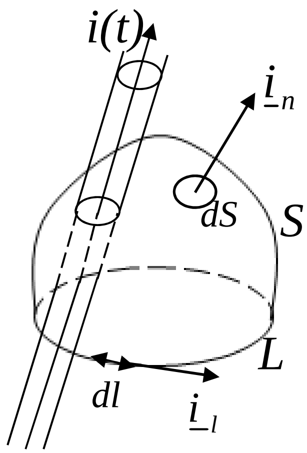
$$\nabla \cdot \underline{j}(\underline{r},t) + \frac{\partial}{\partial t}\left[\nabla \cdot \underline{d}(\underline{r},t)\right] = 0$$

$$\nabla \cdot \underline{d}(\underline{r},t) = \rho(\underline{r},t)$$

$$\Rightarrow \nabla \cdot \underline{j} + \frac{\partial}{\partial t}\rho = 0$$

Continuity equation
(generalization of Kirchhoff's law)

Maxwell’s Equations - Integral form



$$\nabla \times \underline{e} = -\frac{\partial}{\partial t}\underline{b}(\underline{r},t) \quad \& \quad \nabla \times \underline{h}(\underline{r},t) = \frac{\partial}{\partial t}\underline{d}(\underline{r},t) + \underline{j}(\underline{r},t)$$

$$\begin{cases} \iint_S \nabla \times \underline{e} \cdot \underline{i}_n dS = -\iint_S \frac{\partial \underline{b}}{\partial t} \cdot \underline{i}_n dS \\ \iint_S \nabla \times \underline{h} \cdot \underline{i}_n dS = \iint_S \frac{\partial \underline{d}}{\partial t} \cdot \underline{i}_n dS + \iint_S \underline{j} \cdot \underline{i}_n dS \end{cases}$$

displacement current

Stokes’ Theorem: $\oint_L \underline{A} \cdot dl = \iint_S \nabla \times \underline{A} \cdot \underline{i}_n dS$

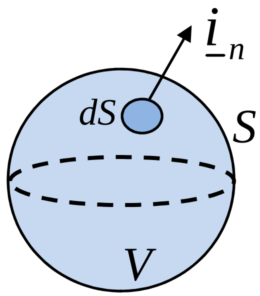
$$\oint_L \underline{e} \cdot \underline{i}_l \cdot dl = -\frac{\partial}{\partial t}\iint_S \underline{b} \cdot \underline{i}_n dS$$

Maxwell-Faraday law

$$\oint_L \underline{h} \cdot \underline{i}_l \cdot dl = \frac{\partial}{\partial t}\iint_S \underline{d} \cdot \underline{i}_n dS + \iint_S \underline{j} \cdot \underline{i}_n dS$$

Generalization of Ampere’s circuital law

Maxwell's Equations Integral form – Gauss's law



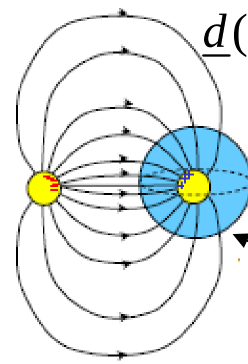
$$\nabla \cdot \underline{d}(\underline{r},t) = \rho(\underline{r},t) \quad \& \quad \nabla \cdot \underline{b}(\underline{r},t) = 0$$

$$\begin{cases} \iiint_V (\nabla \cdot \underline{d})\, dV = \iiint_V \rho\, dV \\ \iiint_V \nabla \cdot \underline{b} \cdot dV = 0 \end{cases} \quad \Rightarrow$$

Gauss Theorem: $\oiint_S \underline{A} \cdot \underline{i}_n dS = \iiint_V \nabla \cdot \underline{A} dV$

$$\Rightarrow \quad \oiint_S \underline{d} \cdot \underline{i}_n dS = \iiint_V \rho dV$$

Gauss's law

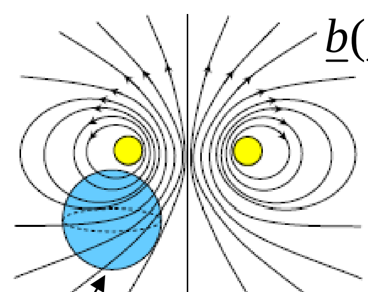


$\underline{d}(\underline{r},t)$

$\oiint_S \underline{d} \cdot \underline{i}_n dS \neq 0$

$$\oiint_S \underline{b} \cdot \underline{i}_n dS = 0$$

Gauss's law for magnetism



$\underline{b}(\underline{r},t)$

$\oiint_S \underline{d} \cdot \underline{i}_n dS = 0$

Maxwell's Equations Integral form – Charge conservation

Continuity equation

$$\nabla \cdot \underline{j} + \frac{\partial}{\partial t}\rho = 0$$

$$\iiint_V (\nabla \cdot \underline{j})\, dV = -\iiint_V \frac{\partial \rho}{\partial t} dV$$

$$\oiint_S \underline{A} \cdot \underline{i}_n dS = \iiint_V \nabla \cdot \underline{A}\, dV$$

$$\Rightarrow \oiint_S \underline{j} \cdot \underline{i}_n dS = -\frac{\partial}{\partial t} \iiint_V \rho\, dV$$

Charge conservation (generalization of Kirchhoff's law)

Maxwell's Equations Integral form – Ampre's law

$$\oint_L \underline{h}\cdot\underline{i}_l\cdot dl = \frac{\partial}{\partial t}\iint_S \underline{d}\cdot\underline{i}_n dS + \iint_S \underline{j}\cdot\underline{i}_n dS$$

$$\phi(t) = \iint_S \underline{d}(\underline{r},t)\cdot\underline{i}_n dS$$

$$i(t) = \oiint_S \underline{j}(\underline{r},t)\cdot\underline{i}_n dS$$

$$\Longrightarrow \oint_L \underline{h}\cdot\underline{i}_l dl = \frac{\partial}{\partial t}\phi(t) + i(t)$$

for static fields: $\frac{\partial}{\partial t}\phi(t) = 0$ $\Longrightarrow$

$$\Longrightarrow \boxed{i(t) = \oint_L \underline{h}\cdot\underline{i}_l dl}$$

Ampere's law (inconsistent for dynamic fields)

$$\nabla\times\underline{h}(\underline{r},t) = \underline{j}(\underline{r},t)$$

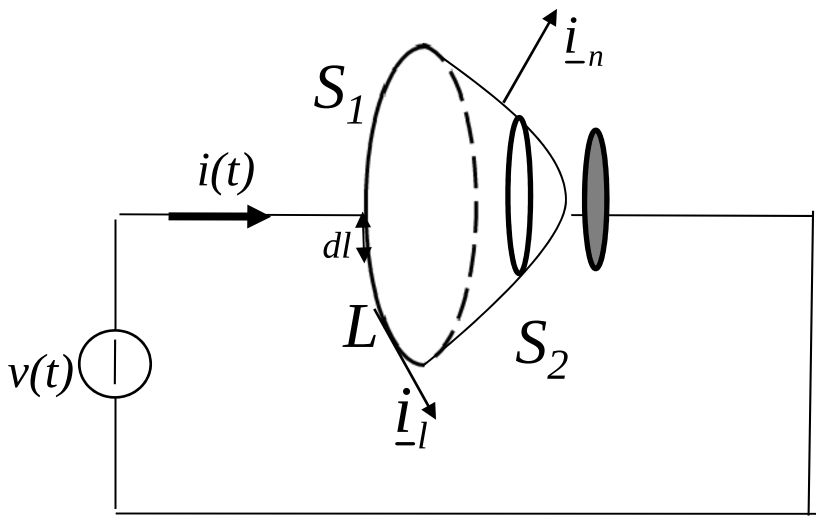
$$\nabla\cdot\nabla\times\underline{h}(\underline{r},t) = \nabla\cdot\underline{j}(\underline{r},t)$$

$$\nabla\cdot(\nabla\times\underline{a}) = 0$$

$$\Longrightarrow \nabla\cdot\underline{j}(\underline{r},t) = 0 \Longrightarrow \iiint_V \nabla\cdot\underline{j}dV = \oiint_S \underline{j}(\underline{r},t)\cdot\underline{i}_n dS = 0$$

but, $i(t) = \oiint_S \underline{j}(\underline{r},t)\cdot\underline{i}_n dS = -\frac{\partial q(t)}{\partial t}$

Maxwell's Equations - Integral form (V)



$$\nabla \times \underline{h}(\underline{r},t) = \underline{j}(\underline{r},t)$$

$$\oint_L \underline{h}(\underline{r},t)\cdot \underline{i}_l dl = \iint_{S_1} \underline{j}(\underline{r},t)\cdot \underline{i}_n dS = i(t)$$

$$\oint_L \underline{h}(\underline{r},t)\cdot \underline{i}_l dl = \iint_{S_2} \underline{j}(\underline{r},t)\cdot \underline{i}_n dS = 0$$

⇨ *Ampere's law gives different results depending on the surface choice!*

Including the second Maxwell's equation in the Ampere's law :

$$\oint_L \underline{h}(\underline{r},t)\cdot \underline{i}_l dl = \iint_{S_1} \underline{j}(\underline{r},t)\cdot \underline{i}_n dS = i(t)$$

$$\oint_L \underline{h}(\underline{r},t)\cdot \underline{i}_l dl = \iint_{S_2} \frac{\partial \underline{d}}{\partial t}\cdot \underline{i}_n dS \neq 0$$

⇨ ***Ampere's generalized law***

$$\oint_L \underline{h}(\underline{r},t)\cdot \underline{i}_l dl = \frac{\partial}{\partial t}\iint_S \underline{d}\cdot \underline{i}_n dS + \iint_S \underline{j}\cdot \underline{i}_n dS$$

Maxwell's Equations - Integral form (VI)

$$\oint_L \underline{e}\cdot\underline{i}_l\cdot dl = -\frac{\partial}{\partial t}\iint_S \underline{b}\cdot\underline{i}_n dS$$

Maxwell-Faraday law

(The induced electromotive force (EMF) in any closed circuit is equal to the time rate of change of the magnetic flux through the circuit.)

$$\oint_L \underline{h}\cdot\underline{i}_l\cdot dl = \frac{\partial}{\partial t}\iint_S \underline{d}\cdot\underline{i}_n dS + \iint_S \underline{j}\cdot\underline{i}_n dS$$

Gneralized Ampere's law

(It relates magnetic fields to electric currents that produce them. Using Ampere's law, one can determine the magnetic field associated with a given current or current associated with a given magnetic field, providing there is no time changing electric field present.)

$$\oiint_S \underline{d}\cdot\underline{i}_n dS = \iiint_V \rho dV$$

Gauss's law

(The electric flux through any closed surface is proportional to the enclosed electric charge.)

$$\oiint_S \underline{b}\cdot\underline{i}_n dS = 0$$

Gauss's law for magnetism

(It states that magnetic charges do not exist.)

$$\oiint_S \underline{j}\cdot\underline{i}_n dS = -\frac{\partial}{\partial t}\iiint_V \rho dV$$

Continuity equation

(The net current through a volume must necessarily equal the net change in charge within the volume.)

Maxwell's Equations – Frequency Domain

Maxwell's equations in frequency domain are formally obtained from Maxwell's equation in time domain by replacing the differential operator $\partial / \partial t$ with $j\omega$:

Differential form:

$$\nabla \times \underline{e}(\underline{r},t) = -\frac{\partial}{\partial t}\underline{b}(\underline{r},t)$$
$$\nabla \times \underline{h}(\underline{r},t) = \frac{\partial}{\partial t}\underline{d}(\underline{r},t) + \underline{j}(\underline{r},t)$$
$$\nabla \cdot \underline{d}(\underline{r},t) = \rho(\underline{r},t)$$
$$\nabla \cdot \underline{b}(\underline{r},t) = 0$$
$$\nabla \cdot \underline{j}(\underline{r},t) + \frac{\partial}{\partial t}\rho(\underline{r},t) = 0$$

⇒

$$\nabla \times \underline{E}(\underline{r},\omega) = -j\omega\underline{B}(\underline{r},\omega)$$
$$\nabla \times \underline{H}(\underline{r},\omega) = j\omega\underline{D}(\underline{r},\omega) + \underline{J}(\underline{r},\omega)$$
$$\nabla \cdot \underline{D}(\underline{r},\omega) = \rho(\underline{r},\omega)$$
$$\nabla \cdot \underline{B}(\underline{r},\omega) = 0$$
$$\nabla \cdot \underline{J}(\underline{r},\omega) + j\omega\rho(\underline{r},\omega) = 0$$

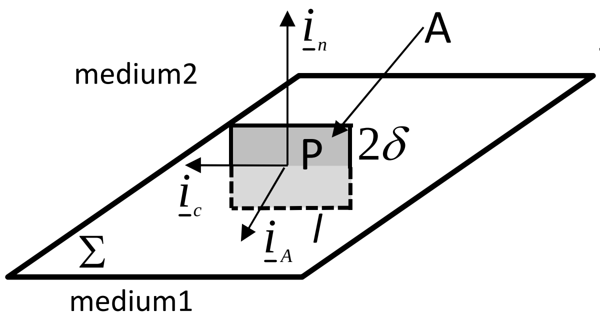
Integral form:

$$\oint_L \underline{e}\cdot\underline{i}_l \cdot dl = -\frac{\partial}{\partial t}\iint_S \underline{b}\cdot\underline{i}_n dS$$
$$\oint_L \underline{h}\cdot\underline{i}_l \cdot dl = \frac{\partial}{\partial t}\iint_S \underline{d}\cdot\underline{i}_n dS + \iint_S \underline{j}\cdot\underline{i}_n dS$$
$$\oiint_S \underline{d}\cdot\underline{i}_n dS = \iiint_V \rho dV$$
$$\oiint_S \underline{b}\cdot\underline{i}_n dS = 0$$
$$\oiint_S \underline{j}\cdot\underline{i}_n dS = -\frac{\partial}{\partial t}\iiint_V \rho dV$$

⇒

$$\oint_L \underline{E}\cdot\underline{i}_l \cdot dl = -j\omega\iint_S \underline{B}\cdot\underline{i}_n dS$$
$$\oint_L \underline{H}\cdot\underline{i}_l \cdot dl = j\omega\iint_S \underline{D}\cdot\underline{i}_n dS + \iint_S \underline{J}\cdot\underline{i}_n dS$$
$$\oiint_S \underline{D}\cdot\underline{i}_n dS = \iiint_V \rho dV$$
$$\oiint_S \underline{B}\cdot\underline{i}_n dS = 0$$
$$\oiint_S \underline{J}\cdot\underline{i}_n dS = -j\omega\iiint_V \rho dV$$

Boundary conditions (I)



Considerations:
- l and δ are small enough.
- $\underline{i}_c \times \underline{i}_A = \underline{i}_n$
- $\oint_L \underline{e} \cdot \underline{i}_l \cdot dl = -\frac{\partial}{\partial t} \iint_S \underline{b} \cdot \underline{i}_n dS$

$$\left(\underline{e}_2 \cdot \underline{i}_c\right) l + \left(\underline{e}_1 \cdot \left(-\underline{i}_c\right)\right) l = -\frac{\partial}{\partial t}\left(l\delta \underline{b}_1\right) - \frac{\partial}{\partial t}\left(l\delta \underline{b}_2\right)$$

If $\delta \to \infty \Rightarrow \underline{i}_c \cdot \underline{e}_2 = \underline{i}_c \cdot \underline{e}_1 \forall \underline{i}_c$, $\underline{i}_c = \underline{i}_A \times \underline{i}_n$

$$\Rightarrow \underline{e}_2 \cdot \left(\underline{i}_A \times \underline{i}_n\right) = \underline{e}_1 \cdot \left(\underline{i}_A \times \underline{i}_n\right) \Leftrightarrow \underline{i}_A \cdot \left(\underline{i}_n \times \underline{e}_2\right) = \underline{i}_A \cdot \left(\underline{i}_n \times \underline{e}_1\right) \Rightarrow \boxed{\left(\underline{i}_n \times \underline{e}_2\right) = \left(\underline{i}_n \times \underline{e}_1\right)}$$

$$\left(\underline{e}_2\right)_{tg} = \left(\underline{e}_1\right)_{tg}$$

From second Maxwell equation $\oint_L \underline{h} \cdot \underline{i}_l \cdot dl = \frac{\partial}{\partial t} \iint_S \underline{d} \cdot \underline{i}_n dS + \iint_S \underline{j} \cdot \underline{i}_n dS$ (and the same considerations):

$$\underline{i}_A \cdot \left(\underline{i}_n \times \left(\underline{h}_2 - \underline{h}_1\right)\right) = \lim_{\delta \to 0}\left(\underline{j} \cdot \underline{i}_A\right)\delta \forall \underline{i}_A \Rightarrow \underline{i}_n \times \left(\underline{h}_2 - \underline{h}_1\right) = \underline{j}_S$$

surface current

$\nexists$ surface current:

$$\left(\underline{i}_n \times \underline{e}_2\right) = \left(\underline{i}_n \times \underline{e}_1\right)$$
$$\left(\underline{i}_n \times \underline{h}_2\right) = \left(\underline{i}_n \times \underline{h}_1\right)$$

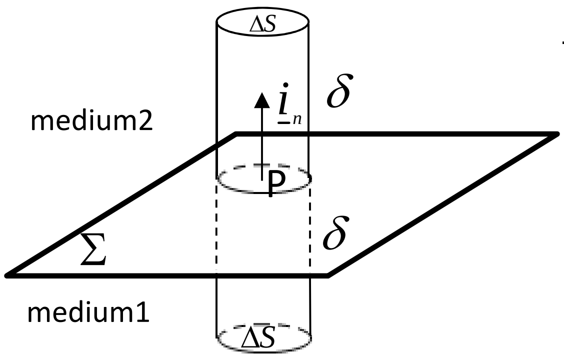
$\exists$ surface current:

$$\left(\underline{i}_n \times \underline{e}_2\right) = \left(\underline{i}_n \times \underline{e}_1\right)$$
$$\underline{i}_n \times \left(\underline{h}_2 - \underline{h}_1\right) = \underline{j}_S$$

→ ***Boundary conditions for the tangential electric and tangential magnetic field***

The tangential electric field component is continuous, while the tangential magnetic field component is discontinuous if the surface current is non zero ($\sigma \to \infty$ PEC), while it is continuous if the surface current is zero.

Boundary conditions (II)



Considerations: l and δ are small enough to consider

$$\oint\!\!\oint_S \underline{b}\cdot\underline{i}_n dS = 0$$

If δ & ΔS are small enough $\Rightarrow \underline{b}_2\cdot\underline{i}_n\Delta S - \underline{b}_1\cdot\underline{i}_n\Delta S = 0 \Rightarrow$ $\boxed{\underline{b}_2\cdot\underline{i}_n = \underline{b}_1\cdot\underline{i}_n}$ → *Boundary conditions for the normal component of magnetic induction*

For the third Maxwell equation $\oint\!\!\oint_S \underline{d}\cdot\underline{i}_n dS = \iiint_V \rho dV$ we assume no electrical surface currents density:

$\Rightarrow$ $\boxed{\underline{d}_2\cdot\underline{i}_n = \underline{d}_1\cdot\underline{i}_n}$ → *Boundary conditions for the normal component of electric induction*

If $\boxed{\exists \text{ surface current}}$ $\Rightarrow$ $\boxed{\underline{i}_n\cdot(\underline{d}_2 - \underline{d}_1) = \lim_{\delta\to 0}\rho\delta = \rho_S}$

If $\sigma\to\infty$ the normal component of the electric induction is discontinuous.

The boundary conditions are also valid in frequency domain – vectors can be replaced by the corresponding phasors.

References

1. G. Manara, A. Monorchio, P.Nepa, “*Appunti di Campi Elettromagnetici*”
2. J. Slater, N. Frank, “*Electromagnetism*”
3. M. Schwartz, “*Principle of Electrodynamics*”
4. http://www.wikipedia.org/