UNIVERSITÀ DI PISA

Electromagnetic Radiations and Biological Interactions

"Laurea Magistrale" in Biomedical Engineering
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Plane waves

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Lecture Content

➢ **Plane waves**

- **Plane Waves in Time Domain**
- **Polarization**
- **Plane Waves in Frequency Domain (in dispersive and lossy media)**

Introduction

- One of the most important consequences of Maxwell's equations is the existence of electric and magnetic field *perturbations* that travels with a *finite* velocity (in a material or even in free-space).

- A particular simple solution of Maxwell's equations is the plane wave solution; it allows introducing the fundamental parameters of electromagnetic wave propagation:

a. propagation constant, phase and attenuation constants
b. wavelength
c. phase velocity
d. medium characteristic impedance
e. polarization

- Plane waves are particularly important in applications where the electromagnetic field distribution, sufficiently far away from the source, can be effectively approximated by a *local* plane wave.

Plane Waves (time domain)

Assumptions: free-space medium (linear, isotropic, homogeneous and non dispersive) with no charges (***ρ=0***) and no currents (***j=0***). A non-vanishing solution can be obtained due to the presence of sources located outside the volume where Maxwell's equations are going to be solved.

$$\varepsilon_0 = 1/36\pi \cdot 10^{-9}\, F/m = 8.854 \cdot 10^{-12}\, F/m$$

$$\mu_0 = 4\pi \cdot 10^{-7}\, H/m$$

$$\begin{cases} \underline{d}(\underline{r},t) = \varepsilon_0 \underline{e}(\underline{r},t) \;\&\; \underline{b}(\underline{r},t) = \mu_0 \underline{h}(\underline{r},t) \\ \nabla \times \underline{e}(\underline{r},t) = -\mu_0 \dfrac{\partial}{\partial t} \underline{h}(\underline{r},t) \\ \nabla \times \underline{h}(\underline{r},t) = \varepsilon_0 \dfrac{\partial}{\partial t} \underline{e}(\underline{r},t) \\ \nabla \cdot [\varepsilon_0 \underline{e}(\underline{r},t)] = 0 \\ \nabla \cdot [\mu_0 \underline{h}(\underline{r},t)] = 0 \end{cases}$$

$\Longrightarrow$

A further assumption: electric and magnetic fields are independent of *x* and *y* coordinates (looking for a solution only dependent on *z*: PLANE WAVE SOLUTION)

$$\underline{e}(z,t) \,\&\, \underline{h}(z,t) \Longrightarrow \nabla \times \underline{e} = \begin{vmatrix} \underline{i}_x & \underline{i}_y & \underline{i}_z \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ e_x & e_y & e_z \end{vmatrix} = -\frac{\partial e_y}{\partial z}\underline{i}_x + \frac{\partial e_x}{\partial z}\underline{i}_y$$

$$\nabla \times \underline{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\underline{i}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\underline{i}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\underline{i}_z$$

Plane Waves (time domain)

$$\left.\begin{aligned}\nabla\times\underline{e}(z,t)&=-\mu_0\frac{\partial}{\partial t}\underline{h}(z,t)\\ \nabla\times\underline{h}(z,t)&=\varepsilon_0\frac{\partial}{\partial t}\underline{e}(z,t)\end{aligned}\right|\Rightarrow \underbrace{\begin{cases}-\dfrac{\partial e_y}{\partial z}\underline{i}_x+\dfrac{\partial e_x}{\partial z}\underline{i}_y=-\mu_0\dfrac{\partial}{\partial t}\left(h_x\underline{i}_x+h_y\underline{i}_y+h_z\underline{i}_z\right)\\ -\dfrac{\partial h_y}{\partial z}\underline{i}_x+\dfrac{\partial h_x}{\partial z}\underline{i}_y=\varepsilon_0\dfrac{\partial}{\partial t}\left(e_x\underline{i}_x+e_y\underline{i}_y+e_z\underline{i}_z\right)\end{cases}}_{\text{TWO } \textbf{vector} \text{ equations}}$$

$$\Rightarrow \underbrace{\begin{cases}\dfrac{\partial e_y}{\partial z}=\mu_0\dfrac{\partial h_x}{\partial t}\\ \dfrac{\partial h_x}{\partial z}=\varepsilon_0\dfrac{\partial e_y}{\partial t}\end{cases}\ \&\ \begin{cases}\dfrac{\partial e_x}{\partial z}=-\mu_0\dfrac{\partial h_y}{\partial t}\\ -\dfrac{\partial h_y}{\partial z}=\varepsilon_0\dfrac{\partial e_x}{\partial t}\end{cases}\ \&\ \begin{cases}\dfrac{\partial e_z}{\partial t}=0\\ \dfrac{\partial h_z}{\partial t}=0\end{cases}}_{\text{SIX } \textbf{scalar} \text{ equations}}$$

$$e_z=const\ \&\ h_z=const$$ *(static components)*

$$we\ will\ assume\ e_z=0\ \&\ h_z=0$$ *(since looking for a dynamic solution)*

Wave equation (D'Alambert's equation)

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

$$\begin{cases} \dfrac{\partial e_x}{\partial z} = -\mu_0 \dfrac{\partial h_y}{\partial t} \\ -\dfrac{\partial h_y}{\partial z} = \varepsilon_0 \dfrac{\partial e_x}{\partial t} \end{cases} \xrightarrow[\frac{\partial}{\partial t}]{\frac{\partial}{\partial z}} \begin{cases} \dfrac{\partial^2 e_x}{\partial z^2} = -\mu_0 \dfrac{\partial^2 h_y}{\partial z \partial t} \\ -\dfrac{\partial^2 h_y}{\partial t \partial z} = \varepsilon_0 \dfrac{\partial^2 e_x}{\partial t^2} \end{cases} \Rightarrow \boxed{\dfrac{\partial^2 e_x}{\partial z^2} = \varepsilon_0 \mu_0 \dfrac{\partial^2 e_x}{\partial t^2}} \rightarrow \Rightarrow \dfrac{\partial^2 e_x}{\partial z^2} = \dfrac{1}{c^2} \dfrac{\partial^2 e_x}{\partial t^2}$$

$$\begin{cases} \dfrac{\partial e_x}{\partial z} = -\mu_0 \dfrac{\partial h_y}{\partial t} \\ -\dfrac{\partial h_y}{\partial z} = \varepsilon_0 \dfrac{\partial e_x}{\partial t} \end{cases} \xrightarrow[\frac{\partial}{\partial z}]{\frac{\partial}{\partial t}} \begin{cases} \dfrac{\partial^2 e_x}{\partial t \partial z} = -\mu_0 \dfrac{\partial^2 h_y}{\partial t^2} \\ -\dfrac{\partial^2 h_y}{\partial z^2} = \varepsilon_0 \dfrac{\partial^2 e_x}{\partial z \partial t} \end{cases} \Rightarrow \boxed{\dfrac{\partial^2 h_y}{\partial z^2} = \varepsilon_0 \mu_0 \dfrac{\partial^2 h_y}{\partial t^2}} \rightarrow \Rightarrow \dfrac{\partial^2 h_y}{\partial z^2} = \dfrac{1}{c^2} \dfrac{\partial^2 h_y}{\partial t^2}$$

$$\begin{cases} \dfrac{\partial^2 e_x}{\partial z^2} - \dfrac{1}{c^2} \dfrac{\partial^2 e_x}{\partial t^2} = 0 \\ \dfrac{\partial^2 h_y}{\partial z^2} - \dfrac{1}{c^2} \dfrac{\partial^2 h_y}{\partial t^2} = 0 \end{cases}$$

<u>D'Alambert's equation</u>

Wave equation (D'Alambert's equation)

$$\begin{cases} \dfrac{\partial^2 e_x}{\partial z^2} - \dfrac{1}{c^2}\dfrac{\partial^2 e_x}{\partial t^2} = 0 \\ \dfrac{\partial^2 h_y}{\partial z^2} - \dfrac{1}{c^2}\dfrac{\partial^2 h_y}{\partial t^2} = 0 \end{cases} \Rightarrow \begin{cases} e_x(z,t) = \boxed{f_1(z-ct)} + \boxed{f_2(z+ct)} \\ h_y(z,t) = \boxed{f_3(z-ct)} + \boxed{f_4(z+ct)} \end{cases}$$

Backward wave (z<0)

Forward wave (z>0)

If considering *only* the forward wave which is function of *u=z-ct*

$$\frac{\partial h_y}{\partial z} = -\varepsilon_0 \frac{\partial e_x}{\partial t} \leftrightarrow \frac{\partial h_y}{\partial u}\frac{\partial u}{\partial z} = -\varepsilon_0 \frac{\partial e_x}{\partial u}\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial z} = 1 \ \& \ \frac{\partial u}{\partial t} = -c$$

$$\Rightarrow \frac{\partial h_y}{\partial u} = \varepsilon_0 c \frac{\partial e_x}{\partial u} \Rightarrow \frac{\partial}{\partial u}\left[h_y - \sqrt{\frac{\varepsilon_0}{\mu_0}} e_x\right] = 0$$

$$\Downarrow$$

$$h_y(u) - \sqrt{\frac{\varepsilon_0}{\mu_0}} e_x(u) = const.\ w.r.t.\ u$$

Due to the finite propagation velocity, the electromagnetic field will be zero at $t=t_0$ in an interval $[z_1,z_2]$ far form the source (if the latter is excited at $t=t_0$); then when *u* is in the interval $[u_1,u_2]=[z_1-ct_0, z_2-ct_0]$ field components are zero and above constant must be zero for any value of *u*.

$$h_y - \sqrt{\frac{\varepsilon_0}{\mu_0}} e_x = const = 0 \quad e_x(z-ct) = \sqrt{\frac{\mu_0}{\varepsilon_0}} h_y(z-ct) = \zeta_0 h_y(z-ct)$$

$$\boxed{\zeta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi\Omega \cong 377\Omega}$$

Free-space characteristic impedance

Plane Waves – properties (forward wave)

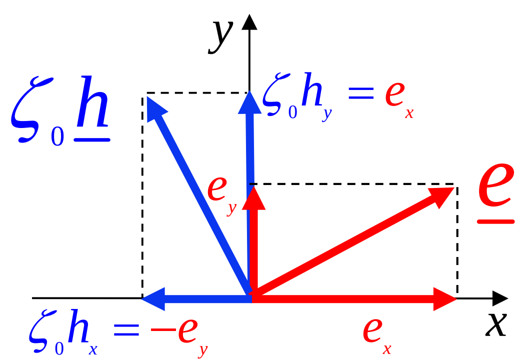
$$\begin{cases} \dfrac{\partial e_x}{\partial z} = -\mu_0 \dfrac{\partial h_y}{\partial t} \\ -\dfrac{\partial h_y}{\partial z} = \varepsilon_0 \dfrac{\partial e_x}{\partial t} \end{cases} \quad \& \; z>0 \quad \textit{(forward wave)}$$

$$\begin{cases} \dfrac{\partial e_y}{\partial z} = \mu_0 \dfrac{\partial h_x}{\partial t} \\ \dfrac{\partial h_x}{\partial z} = \varepsilon_0 \dfrac{\partial e_y}{\partial t} \end{cases} \quad \& \; z>0 \quad \textit{(forward wave)}$$

$$\Longrightarrow$$

$$e_x(z-ct) = \sqrt{\frac{\mu_0}{\varepsilon_0}}\, h_y(z-ct) = \zeta_0 h_y(z-ct)$$

$$e_y(z-ct) = -\sqrt{\frac{\mu_0}{\varepsilon_0}}\, h_x(z-ct) = -\zeta_0 h_x(z-ct)$$



$$\boxed{\frac{e_x(z-ct)}{h_y(z-ct)} = -\frac{e_y(z-ct)}{h_x(z-ct)} = \zeta_0}$$

The instantaneous electric field can vary arbitrarily in time while magnetic field vector (amplitude and direction) will satisfy above conditions at any time and for any observation point

Plane Waves – properties (backward wave)

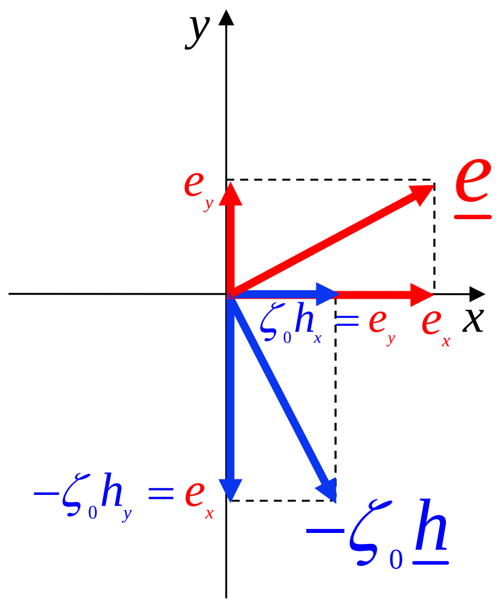
$$\begin{cases} \dfrac{\partial e_x}{\partial z} = -\mu_0 \dfrac{\partial h_y}{\partial t} \\ -\dfrac{\partial h_y}{\partial z} = \varepsilon_0 \dfrac{\partial e_x}{\partial t} \end{cases} \quad \& \quad z<0 \quad \textit{(backward wave)}$$

$$\begin{cases} \dfrac{\partial e_y}{\partial z} = \mu_0 \dfrac{\partial h_x}{\partial t} \\ \dfrac{\partial h_x}{\partial z} = \varepsilon_0 \dfrac{\partial e_y}{\partial t} \end{cases} \quad \& \quad z<0 \quad \textit{(backward wave)}$$

$$\Longrightarrow$$

$$e_x(z+ct) = -\sqrt{\frac{\mu_0}{\varepsilon_0}}\, h_y(z+ct) = -\zeta_0 h_y(z+ct)$$

$$e_y(z+ct) = \sqrt{\frac{\mu_0}{\varepsilon_0}}\, h_x(z+ct) = \zeta_0 h_x(z+ct)$$



$$\boxed{-\frac{e_x(z+ct)}{h_y(z+ct)} = \frac{e_y(z+ct)}{h_x(z+ct)} = \zeta_0}$$

The instantaneous electric field can vary arbitrarily in time while magnetic field vector (amplitude and direction) will satisfy above conditions at any time and for any observation point

Plane Waves – properties

✓ *Electric field and magnetic field are constant on any plane perpendicular to the plane wave propagation direction.*
✓ *The (finite) perturbation propagation velocity is* $c = 1/\sqrt{\varepsilon_0 \mu_0}$
✓ *The electric and magnetic fields are related by the medium characteristic impedance* ζ_0

For each plane wave (forward or backward)

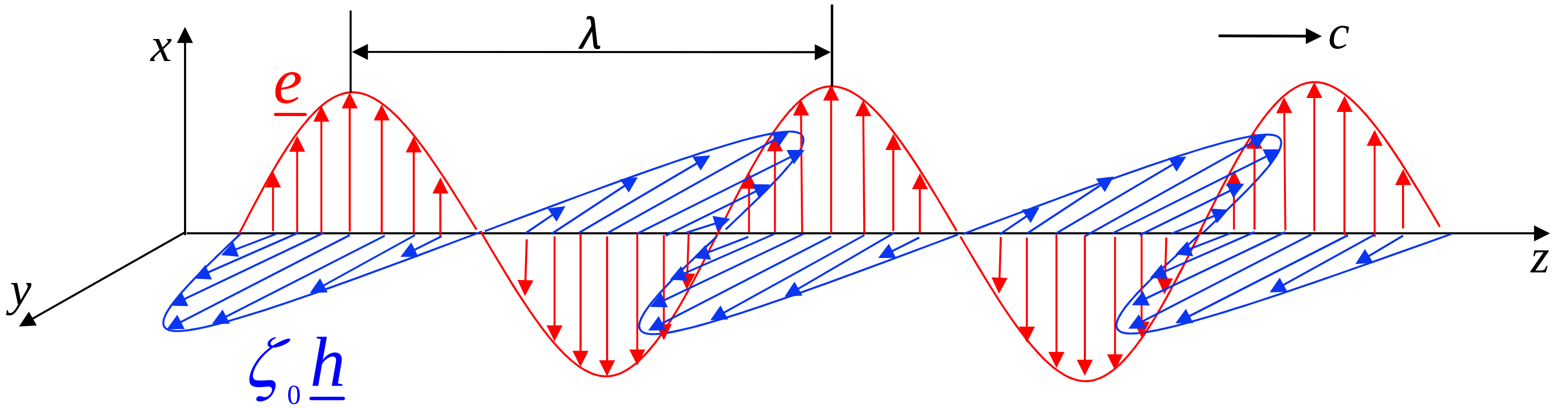
1. $\underline{e} \perp \underline{h}, \forall (z,t) \qquad \left(\underline{e}\cdot\underline{h} = e_x h_x + e_y h_y = \zeta_0 h_y h_x - \zeta_0 h_x h_y = 0\right)$
2. $\dfrac{|\underline{e}|}{|\underline{h}|} = \zeta_0, \forall (z,t) \qquad \left(|\underline{e}| = \sqrt{|e_x|^2 + |e_y|^2} = \sqrt{|\zeta_0 e_x|^2 + |\zeta_0 e_y|^2} = \zeta_0 |\underline{h}|\right)$
3. $$\begin{cases} z > 0\,(\textit{forward wave}) \rightarrow \underline{h} = \dfrac{1}{\zeta_0}\underline{i}_z \times \underline{e} \Rightarrow \underline{e} = \zeta_0 \underline{h} \times \underline{i}_z \\ z < 0\,(\textit{backward wave}) \rightarrow \underline{h} = \dfrac{1}{\zeta_0}(-\underline{i}_z) \times \underline{e} \Rightarrow \underline{e} = \zeta_0 \underline{h} \times (-\underline{i}_z) \end{cases}$$

The physical properties of a plane wave are independent of the coordinate system and propagation direction. For any propagation direction denoted by the versor $\underline{i}$*:*

$$\forall\left(\underline{r},t\right): \quad \underline{e}(\underline{r}\cdot\underline{i} - c\,t)\,\&\,\underline{h}(\underline{r}\cdot\underline{i} - c\,t), \quad \underline{e}\cdot\underline{i} = 0 \quad \underline{h} = \frac{1}{\zeta_0}\underline{i}\times\underline{e}$$

or equivalently $\quad \underline{h}\cdot\underline{i} = 0 \quad \underline{e} = \zeta_0 \underline{h}\times\underline{i}$

Plane Waves – example: sinusoidal signals



$$e_x = b\cos\left[\frac{2\pi}{\lambda}(z-ct)\right]$$

$$e_y = 0; e_z = 0$$

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$$

$\Rightarrow$

$$e_x = b\cos\left[\frac{2\pi}{\lambda}(z-ct)\right] = b\cos\left[\frac{2\pi}{\lambda}z - \frac{2\pi}{T}t\right] = b\cos(\omega t - \beta z)$$

$T = \lambda / c \longrightarrow$ time period $\quad \omega = 2\pi / T = 2\pi f$

$\beta = \dfrac{2\pi}{\lambda} \longrightarrow$ phase constant

$$e_x = b\cos(\omega t - \beta z) \rightarrow \begin{cases} z = 0 \rightarrow e_x = b\cos(\omega t) \\ z = \dfrac{\lambda}{4} = \dfrac{2\pi}{\beta}\cdot\dfrac{1}{4} \rightarrow e_x = b\cos(\omega t - \pi/2) = b\sin(\omega t) \end{cases}$$

For magnetic field:

$$h_x = 0; h_y = -\zeta_0 e_y; h_z = 0$$

$$\boxed{\lambda = \frac{2\pi}{\beta} = \frac{c}{f} \quad \text{Electromagnetic wavelength = spatial period}}$$

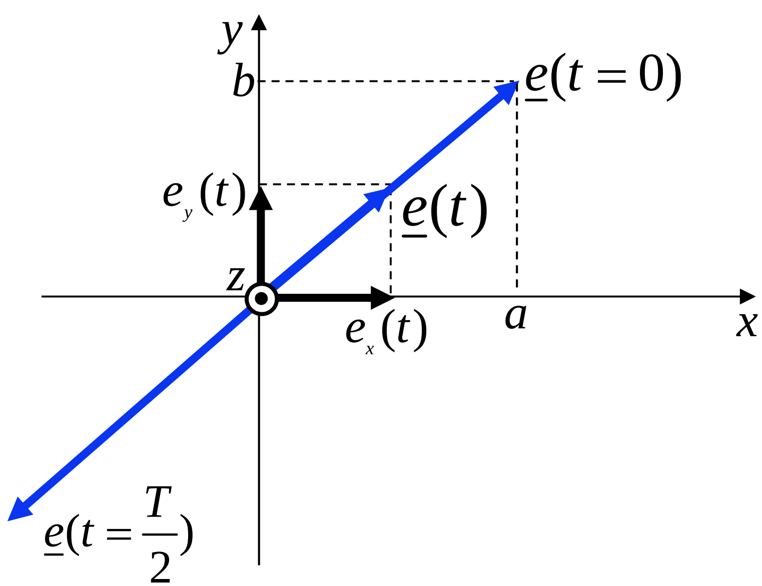
Linear Polarization

Polarization=determines the orientation of the electric field in a fixed spatial plane orthogonal to the direction of the propagation.

Assume $z=0$ $\begin{cases} e_x = a\cos(\omega t - \beta z) = a\cos(\omega t) \\ e_y = b\cos(\omega t - \beta z + \delta) = b\cos(\omega t + \delta) \end{cases} \Rightarrow \underline{e} = e_x \underline{i}_x + e_y \underline{i}_y = a\cos(\omega t)\underline{i}_x + b\cos(\omega t + \delta)\underline{i}_y$

$$\boxed{\delta = 0} \rightarrow$$ **Linear Polarization**

If e_x and e_y are in phase $\Rightarrow$ $\underline{e}$ is linearly polarized along a direction given by the angle: $\alpha = tg^{-1}\left(\frac{e_y}{e_x}\right) = tg^{-1}\frac{b}{a}$

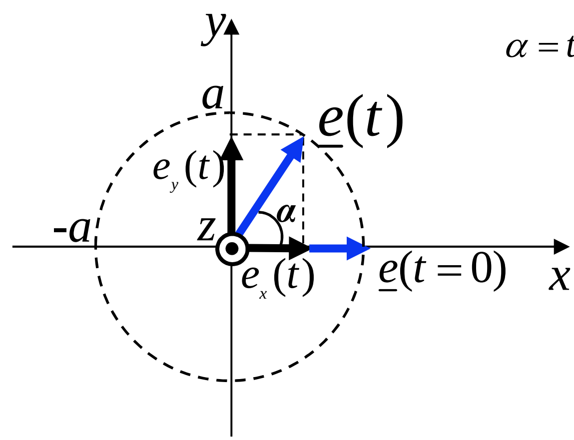


Circular Polarization

$$\boxed{a = b \;\&\; \delta = \pm\pi/2} \rightarrow \text{Circular Polarization}$$

$$\underline{e} = a\cos(\omega t)\underline{i}_x + a\cos\left(\omega t \pm \frac{\pi}{2}\right)\underline{i}_y = a\cos(\omega t)\underline{i}_x \pm a\sin(\omega t)\underline{i}_y \;\Rightarrow\; |\underline{e}| = a \quad \forall t$$

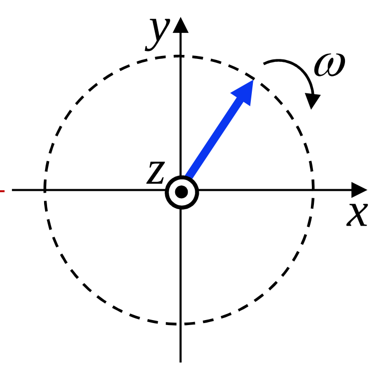
$$\alpha = tg^{-1}\left(\frac{e_y}{e_x}\right) = tg^{-1}\left(\mp\frac{\sin\omega t}{\cos\omega t}\right) = \mp\omega t$$

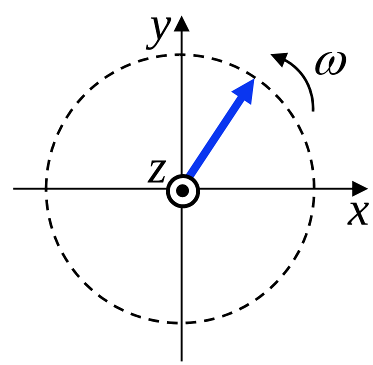


$$\boxed{\delta = \pi/2}$$

$$\boxed{\delta = -\pi/2}$$

Light Hand Circular Polarization (LHCP)





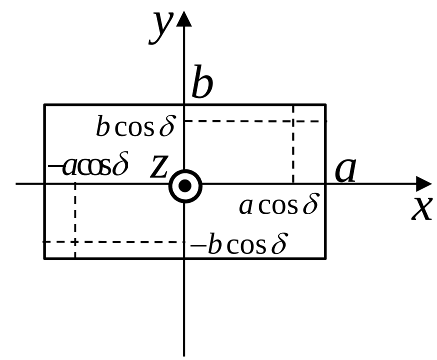
Right Hand Circular Polarization (RHCP)

Elliptical Polarization

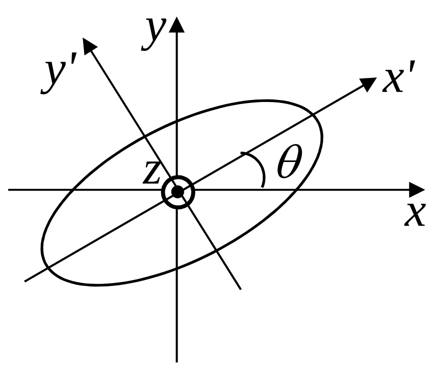
$$\boxed{a \neq b \;\&\; \delta \neq \pm\pi/2} \longrightarrow \text{Elliptical Polarization}$$

Consider $\begin{cases} e_x = a\cos(\omega t) \\ e_y = b\cos(\omega t+\delta) \end{cases} \Rightarrow \left(\dfrac{e_x}{a}\right)^2 + \left(\dfrac{e_y}{b}\right)^2 = \cos^2\omega t + \cos^2(\omega t+\delta) \xrightarrow{-\cos\omega t} \left(\dfrac{e_x}{a}\right)^2 + \left(\dfrac{e_y}{b}\right)^2 - 2\cos\delta\dfrac{e_x}{a}\dfrac{e_y}{b} - \sin^2\delta = 0$

If: $\begin{cases} e_x = a \Rightarrow e_y = b\cos\delta \\ e_x = -a \Rightarrow e_y = -b\cos\delta \\ e_y = b \Rightarrow e_x = a\cos\delta \\ e_y = -b \Rightarrow e_x = -a\cos\delta \\ |e_x| < a \;\&\; |e_y| < b \end{cases} \Longrightarrow$



If the system xy is rotated by a θ angle: $tg2\theta = \dfrac{2ab}{a^2-b^2}\cos\delta \Rightarrow \boxed{\left(\dfrac{e_{x'}}{a'}\right)^2 + \left(\dfrac{e_{y'}}{b'}\right)^2 = 1} \longrightarrow$ *Ellipse equation*



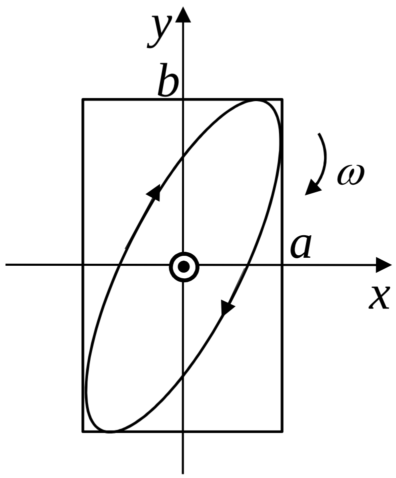
$$\alpha = \angle(\underline{e}, x);\; tg\alpha = \frac{e_y}{e_x} = \frac{b\cos(\omega t+\delta)}{a\cos(\omega t)}$$

Angular velocity: $\omega(t) = \dfrac{d\alpha}{dt} = -\omega\dfrac{ab\sin\delta}{e^2(t)}$

Plane Wave Polarization

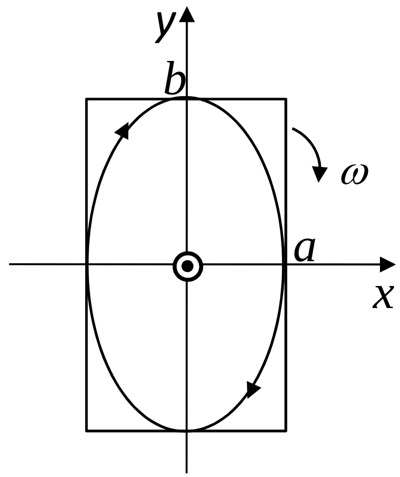
a<b

$\delta = \pi / 4$



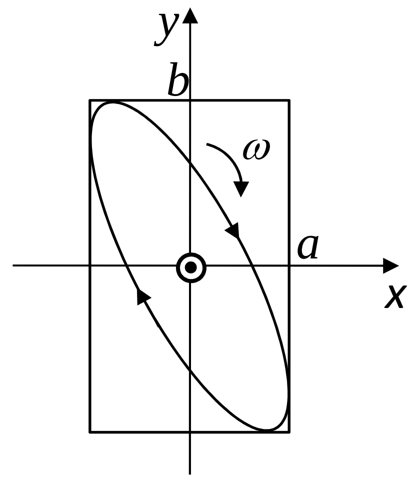
Elliptical polarization (LHEP)

$\delta = \pi / 2$



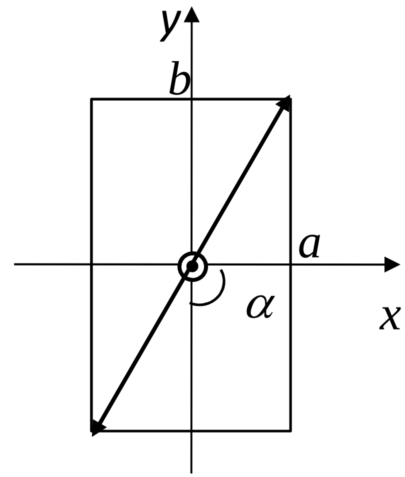
Elliptical polarization (LHEP)

$\delta = 3\pi / 4$



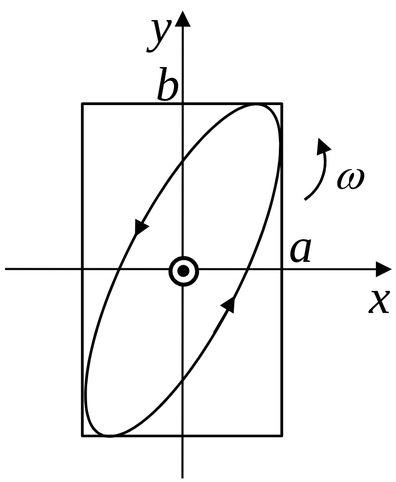
Elliptical polarization (LHEP)

$\delta = 0$



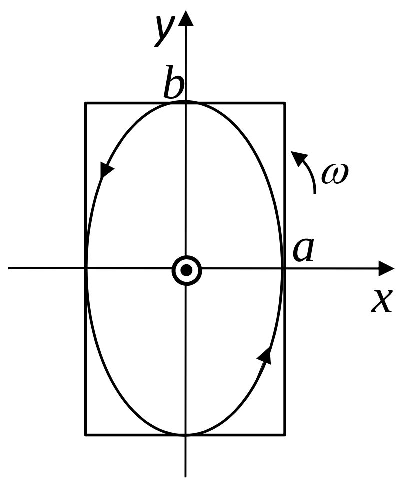
Linear polarization (ellipse becomes a line)

$\delta = -\pi / 4$



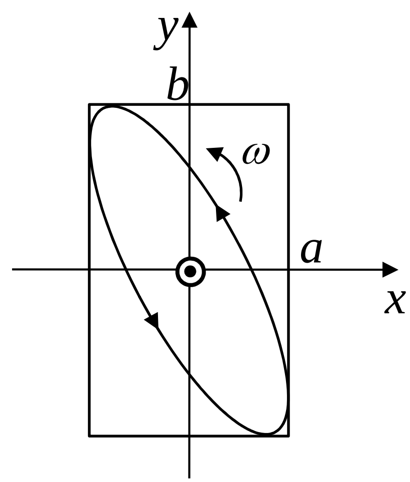
Elliptical polarization (RHEP)

$\delta = -\pi / 2$



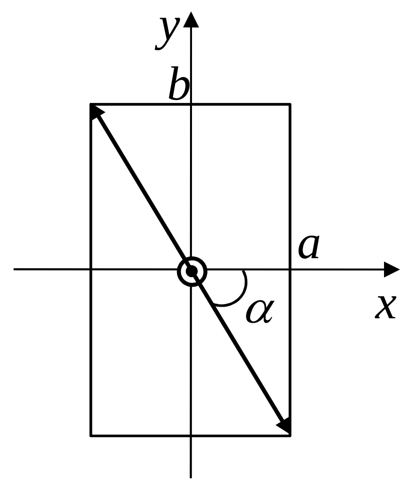
Elliptical polarization (RHEP)

$\delta = -3\pi / 4$



Elliptical polarization (RHEP)

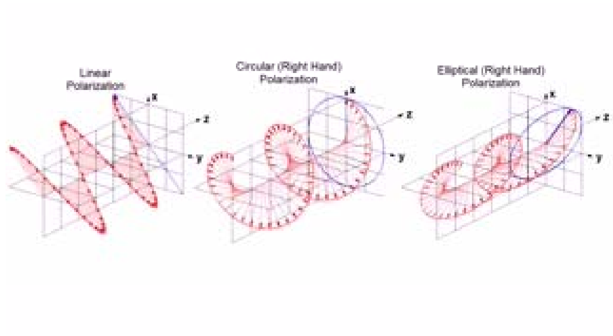
$\delta = -\pi$



Linear polarization (ellipse becomes a line)

Plane Wave Polarization

Linear – Circular- Elliptical Polarization Animation



http://www.youtube.com/watch?v=Q0qrU4nprB0

Plane Waves – Frequency Domain solution

Frequency domain analysis allows to study EM propagation in dissipative and dispersive medium

Medium: linear, homogeneous, isotropic, dispersive in time and with losses

$$\begin{cases} \underline{D}(\underline{r},\omega) = \varepsilon(\omega)\underline{E}(\underline{r},\omega) \\ \underline{B}(\underline{r},\omega) = \mu(\omega)\underline{H}(\underline{r},\omega) \end{cases}$$

In time domain it corresponds to a temporal convolution

$$\begin{cases} \varepsilon(\omega) = \varepsilon'(\omega) - j\varepsilon''(\omega) \\ \mu(\omega) = \mu'(\omega) - j\mu''(\omega) \end{cases}$$

Dependence on ω accounts for time-dispersion, while the imaginary part is related to losses

Frequency domain Maxwell's equations

$$\begin{cases} \nabla \times \underline{E}(\underline{r},\omega) = -j\omega\underline{B}(\underline{r},\omega) \\ \nabla \times \underline{H}(\underline{r},\omega) = j\omega\underline{D}(\underline{r},\omega) + \underline{J}(\underline{r},\omega) \\ \nabla \cdot (\underline{D}(\underline{r},\omega)) = \rho(\underline{r},\omega) \\ \nabla \cdot (\underline{B}(\underline{r},\omega)) = 0 \end{cases} \Rightarrow \begin{cases} \nabla \times \underline{E}(\underline{r},\omega) = -j\omega\mu(\omega)\underline{H}(\underline{r},\omega) \\ \nabla \times \underline{H}(\underline{r},\omega) = j\omega\varepsilon(\omega)\underline{E}(\underline{r},\omega) + \underline{J}(\underline{r},\omega) \\ \nabla \cdot (\varepsilon(\omega)\underline{E}(\underline{r},\omega)) = \rho(\underline{r},\omega) \\ \nabla \cdot (\mu(\omega)\underline{H}(\underline{r},\omega)) = 0 \end{cases}$$

Medium homogeneity $\Rightarrow \begin{cases} \varepsilon(\omega)\nabla \cdot \underline{E}(\underline{r},\omega) = 0 \\ \mu(\omega)\nabla \cdot \underline{H}(\underline{r},\omega) = 0 \end{cases} \Rightarrow \begin{cases} \nabla \cdot \underline{E}(\underline{r},\omega) = 0 \\ \nabla \cdot \underline{H}(\underline{r},\omega) = 0 \end{cases}$

Plane Waves – Frequency Domain solution

<u>*Assumption:*</u> $\underline{J}(\underline{r},\omega)=0\ \&\ \rho(\underline{r},\omega)=0 \Rightarrow \begin{cases} \nabla\times\underline{E}(\underline{r},\omega)=-j\omega\mu(\omega)\underline{H}(\underline{r},\omega) \\ \nabla\times\underline{H}(\underline{r},\omega)=j\omega\varepsilon(\omega)\underline{E}(\underline{r},\omega) \\ \nabla\cdot\underline{E}(\underline{r},\omega)=0 \\ \nabla\cdot\underline{H}(\underline{r},\omega)=0 \end{cases}$

$$\left\{\begin{array}{l} \dfrac{-1}{j\omega\mu(\omega)}\nabla\times\underline{E}(\underline{r},\omega)=\underline{H}(\underline{r},\omega) \\ \dfrac{-1}{j\omega\mu(\omega)}\nabla\times\left(\nabla\times\underline{E}(\underline{r},\omega)\right)=j\omega\varepsilon(\omega)\underline{E}(\underline{r},\omega) \end{array}\right| \Rightarrow$$

$$\left(\nabla^2\underline{A}=\nabla\left(\nabla\cdot\underline{A}\right)-\nabla\times\left(\nabla\times\underline{A}\right)\right)$$

$$\left.\begin{array}{l} -\nabla^2\underline{E}(\underline{r},\omega)+\nabla\left(\cancel{\nabla\cdot\underline{E}(\underline{r},\omega)}\right)=\omega^2\varepsilon(\omega)\mu(\omega)\underline{E}(\underline{r},\omega) \\ \qquad\qquad\qquad\qquad\quad \rightarrow =0 \\ \nabla\cdot\underline{E}(\underline{r},\omega)=0 \end{array}\right| \Rightarrow$$

$$\boxed{\nabla^2\underline{E}(\underline{r},\omega)+\omega^2\varepsilon(\omega)\mu(\omega)\underline{E}(\underline{r},\omega)=0}$$

Plane Waves – Frequency Domain solution

$$\begin{cases} \nabla \times \underline{E}(\underline{r},\omega) = -j\omega\mu(\omega)\underline{H}(\underline{r},\omega) \\ \nabla \times \underline{H}(\underline{r},\omega) = j\omega\varepsilon(\omega)\underline{E}(\underline{r},\omega) \\ \nabla \cdot \underline{E}(\underline{r},\omega) = 0 \\ \nabla \cdot \underline{H}(\underline{r},\omega) = 0 \end{cases}$$

$$\left.\begin{cases} \dfrac{1}{j\omega\varepsilon(\omega)} \nabla \times \underline{H}(\underline{r},\omega) = \underline{E}(\underline{r},\omega) \\ \dfrac{1}{j\omega\varepsilon(\omega)} \nabla \times \left(\nabla \times \underline{H}(\underline{r},\omega)\right) = -j\omega\mu(\omega)\underline{H}(\underline{r},\omega) \end{cases}\right| \Rightarrow$$

$$\left(\nabla^2 \underline{A} = \nabla(\nabla \cdot \underline{A}) - \nabla \times (\nabla \times \underline{A})\right)$$

$$\left.\begin{matrix} -\nabla^2 \underline{H}(\underline{r},\omega) + \nabla\big(\underbrace{\cancel{\nabla \cdot \underline{H}(\underline{r},\omega)}}_{=0}\big) = \omega^2\varepsilon(\omega)\mu(\omega)\underline{H}(\underline{r},\omega) \\ \nabla \cdot \underline{H}(\underline{r},\omega) = 0 \end{matrix}\right| \Rightarrow$$

$$\boxed{\nabla^2 \underline{H}(\underline{r},\omega) + \omega^2\varepsilon(\omega)\mu(\omega)\underline{H}(\underline{r},\omega) = 0}$$

$$\boxed{\begin{matrix} \nabla^2 \underline{E}(\underline{r},\omega) + k^2 \underline{E}(\underline{r},\omega) = 0 \\ \nabla^2 \underline{H}(\underline{r},\omega) + k^2 \underline{H}(\underline{r},\omega) = 0 \end{matrix}}$$

Homogeneous Vector Helmholtz's Equations

$$\boxed{k = \omega\sqrt{\varepsilon(\omega)\mu(\omega)}}$$

Propagation constant

Helmholtz's equation solution

$$\nabla^2 \underline{E}(\underline{r},\omega) + k^2 \underline{E}(\underline{r},\omega) = 0$$

In a rectangular coordinate system: $\nabla^2 \underline{E}(\underline{r},\omega) = \nabla^2 E_x(\underline{r},\omega)\underline{i}_x + \nabla^2 E_y(\underline{r},\omega)\underline{i}_y + \nabla^2 E_z(\underline{r},\omega)\underline{i}_z$

$$\left(\nabla^2 \underline{A} = \nabla\left(\nabla \cdot \underline{A}\right) - \nabla \times \left(\nabla \times \underline{A}\right)\right)$$

Consider only the component along x:

$$\nabla^2 E_x(\underline{r},\omega) + k^2 E_x(\underline{r},\omega) = 0 \Rightarrow \frac{\partial^2 E_x(\underline{r},\omega)}{\partial^2 x} + \frac{\partial^2 E_x(\underline{r},\omega)}{\partial^2 y} + \frac{\partial^2 E_x(\underline{r},\omega)}{\partial^2 z} + k^2 E_x(\underline{r},\omega) = 0$$

$$\left(\nabla^2 \phi = \frac{\partial^2 \phi}{\partial^2 x} + \frac{\partial^2 \phi}{\partial^2 y} + \frac{\partial^2 \phi}{\partial^2 z}\right)$$

$$\boxed{\textit{Assumption}:\ \underline{E}(\underline{r},\omega) = \underline{E}(z,\omega)}$$

(looking for a solution only dependent on *z*: PLANE WAVE SOLUTION)

$$\frac{\partial^2 E_x(z,\omega)}{\partial z^2} + k^2 E_x(z,\omega) = 0 \ \Rightarrow\ E_x(z,\omega) = E_x^+ e^{-jkz} + E_x^- e^{jkz}$$

In a similar way, it can be shown that:

$$E_y(z,\omega) = E_y^+ e^{-jkz} + E_y^- e^{jkz}$$

$$H_x(z,\omega) = H_x^+ e^{-jkz} + H_x^- e^{jkz}$$

$$H_y(z,\omega) = H_y^+ e^{-jkz} + H_y^- e^{jkz}$$

Helmholtz's equation solution

$$\begin{cases} \nabla \times \underline{E}(z,\omega) = -j\omega\mu(\omega)\underline{H}(z,\omega) \\ \nabla \times \underline{H}(z,\omega) = j\omega\varepsilon(\omega)\underline{E}(z,\omega) \\ \nabla \cdot \underline{E}(z,\omega) = 0 \\ \nabla \cdot \underline{H}(z,\omega) = 0 \end{cases}$$

$$\nabla \times \underline{E}(z,\omega) = \begin{vmatrix} i_x & i_y & i_z \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -\frac{\partial E_y}{\partial z}\underline{i}_x + \frac{\partial E_x}{\partial z}\underline{i}_y = -j\omega\mu(\omega)\underline{H}(z,\omega)$$

$$\nabla \times \underline{H}(z,\omega) = \begin{vmatrix} i_x & i_y & i_z \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = -\frac{\partial H_y}{\partial z}\underline{i}_x + \frac{\partial H_x}{\partial z}\underline{i}_y = j\omega\varepsilon(\omega)\underline{E}(z,\omega)$$

$$\Rightarrow E_z = 0 \quad H_z = 0$$

Field components along the propagation direction must vanish

Helmholtz's equation solution

$$\begin{cases} \dfrac{\partial E_x(z,\omega)}{\partial z} = -j\omega\mu H_y(z,\omega) \\ -\dfrac{\partial H_y(z,\omega)}{\partial z} = j\omega\varepsilon E_x(z,\omega) \end{cases} \Rightarrow \begin{matrix} E_x^+ = \zeta H_y^+ \\ E_x^- = -\zeta H_y^- \end{matrix} \Rightarrow$$

$$\boxed{\begin{aligned} E_x(z,\omega) &= E_x^+ e^{-jkz} + E_x^- e^{jkz} \\ H_y(z,\omega) &= \frac{E_x^+}{\zeta} e^{-jkz} - \frac{E_x^-}{\zeta} e^{jkz} \end{aligned}}$$

$$\begin{cases} \dfrac{\partial E_y(z,\omega)}{\partial z} = j\omega\mu H_x(z,\omega) \\ \dfrac{\partial H_x(z,\omega)}{\partial z} = j\omega\varepsilon E_{yx}(z,\omega) \end{cases} \Rightarrow \begin{matrix} E_y^+ = -\zeta H_x^+ \\ E_y^- = \zeta H_x^- \end{matrix} \Rightarrow$$

$$\boxed{\begin{aligned} E_x(z,\omega) &= E_x^+ e^{-jkz} + E_x^- e^{jkz} \\ H_y(z,\omega) &= \frac{E_x^+}{\zeta} e^{-jkz} - \frac{E_x^-}{\zeta} e^{jkz} \end{aligned}}$$

$$\zeta = \sqrt{\mu / \varepsilon} = R + jX$$ *Medium characteristic impedance*

$$k = \omega\sqrt{\varepsilon(\omega)\mu(\omega)} = \beta - j\alpha \quad \begin{cases} \beta\text{- } \underline{\text{Phase constant}} \\ \alpha\text{- } \underline{\text{Attenuation constant}} \end{cases}$$

If $\alpha = 0$ (lossless medium, μ and ε real): $k = \omega\sqrt{\varepsilon\mu} = \beta$

$$\Rightarrow e^{-jkz} = e^{-j(\beta - j\alpha)z} = e^{-j\beta z} e^{-\alpha z}$$

Plane Waves – phase velocity

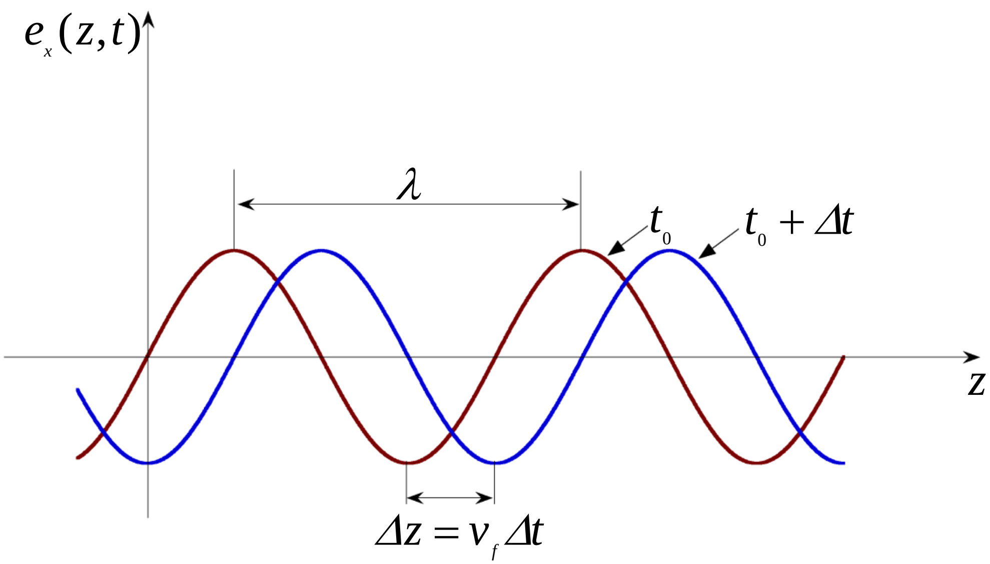
Back to time domain: $e_x(z,t) = \mathrm{Re}\left\{E_x^+ e^{-jkz} e^{j\omega t}\right\} = \mathrm{Re}\left\{\left|E_x^+\right| e^{-\alpha z} e^{j\omega t} e^{-j\beta z} e^{-j\phi_x^+}\right\} = \left|E_x^+\right| e^{-\alpha z}\cos(\omega t - \beta z + \phi_x^+)$

$\phi(z,t) = \omega t - \beta z + \phi_x^+$ (phase of $e_x(z,t)$)

Consider φ at (z,t) and (z+Δz,t+Δt):

$$\Delta\phi = \left[\omega(t_0 + \Delta t) - \beta(z + \Delta z)\right] - (\omega t_0 - \beta z) = \omega\Delta t - \beta\Delta z \qquad \Delta\phi = 0 \Leftrightarrow \frac{\Delta z}{\Delta t} = \frac{\omega}{\beta}$$

$$\boxed{v_f = \frac{\Delta z}{\Delta t} = \frac{\omega}{\beta}}$$

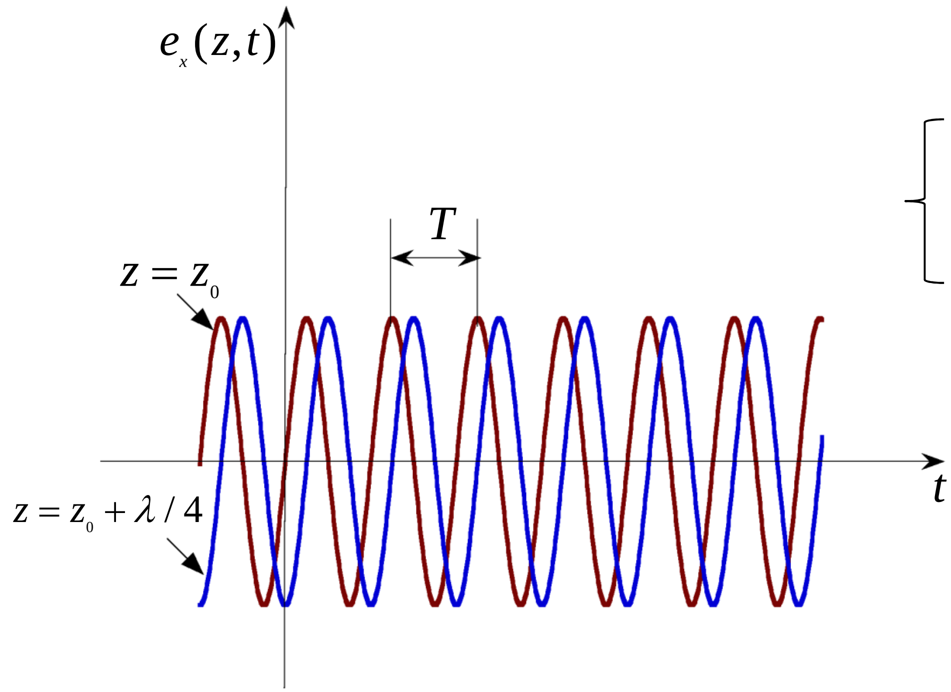


Phase velocity $v_f = \dfrac{\omega}{\beta} = \dfrac{\omega}{\mathrm{Re}\{k\}}$

$\lambda = \dfrac{2\pi}{\beta} = \dfrac{v_f}{f}$ Electromagnetic wavelength = spatial period

Plane Waves – phase velocity

Back to time domain: $$e_x(z,t)=\mathrm{Re}\left\{E_x^+e^{-jkz}e^{j\omega t}\right\}=\mathrm{Re}\left\{\left|E_x^+\right|e^{-\alpha z}e^{j\omega t}e^{-j\beta z}e^{-j\phi_x^+}\right\}=\left|E_x^+\right|e^{-\alpha z}\cos(\omega t-\beta z+\phi_x^+)$$



$\phi(z,t)=\omega t-\beta z+\phi_x^+$ (phase of $e_x(z,t)$)

$$\begin{cases} z=0\rightarrow e_x=b\cos(\omega t) \\ z=\dfrac{\lambda}{4}=\dfrac{2\pi}{\beta}\cdot\dfrac{1}{4}\rightarrow e_x=b\cos(\omega t-\pi/2)=b\sin(\omega t) \end{cases}$$

Free-space $(\varepsilon_0,\mu_0)\Rightarrow$ $$v_f=\frac{1}{\sqrt{\varepsilon_0\mu_0}}=c\cong 3\cdot 10^8\,m/\sec$$

If $\alpha=0$ (lossless medium, μ and ε are real): $$v_f\,\frac{\omega}{\omega\sqrt{\varepsilon(\omega)\mu(\omega)}}=\frac{1}{\sqrt{\varepsilon(\omega)\mu(\omega)}}$$

Plane Waves – properties

$$E_x(z,\omega) = E_x^+ e^{-jkz} + E_x^- e^{jkz}$$

$$E_x^+ = \zeta H_y^+ \quad E_x^- = -\zeta H_y^-$$

$$H_y(z,\omega) = E_x^+ / \zeta e^{-jkz} - E_x^- / \zeta e^{jkz}$$

$$E_y(z,\omega) = E_y^+ e^{-jkz} + E_y^- e^{jkz}$$

$$E_y^+ = -\zeta H_x^+ \quad E_y^- = \zeta H_x^-$$

$$H_x(z,\omega) = -E_y^+ / \zeta e^{-jkz} + E_y^- / \zeta e^{jkz}$$

$$E_z = 0 \;\; H_z = 0 \qquad \zeta = \sqrt{\mu / \varepsilon} = R + jX$$

$$k = \omega\sqrt{\varepsilon(\omega)\mu(\omega)} = \beta - j\alpha \quad \begin{cases} \beta\text{- } \underline{\textit{Phase constant}} & \text{[rad/m]} \\ \alpha\text{- } \underline{\textit{Attenuation constant}} & \text{[m}^{-1}\text{] or [Neper/m]} \end{cases}$$

1. $\underline{E} \bullet \underline{H} = 0$
2. $\dfrac{|\underline{E}|}{|\underline{H}|} = |\zeta|$
3. $\begin{cases} z > 0\,(\,forward\ wave) \rightarrow \underline{H} = \dfrac{1}{\zeta}\underline{i}_z \times \underline{E} \Rightarrow E = \zeta \underline{H} \times \underline{i}_z \\ z < 0\,(backward\ wave) \rightarrow \underline{H} = \dfrac{1}{\zeta}(-\underline{i}_z) \times \underline{E} \Rightarrow E = \zeta \underline{H} \times (-\underline{i}_z) \end{cases}$

Dielectric medium with μ and ε real:

$$\beta = \omega\sqrt{\varepsilon_0 \varepsilon_r \mu_0} \qquad \lambda = \frac{2\pi}{\beta} = \frac{c}{f\sqrt{\varepsilon_r}} = \frac{\lambda_0}{\sqrt{\varepsilon_r}} \qquad v_f = \frac{c}{\sqrt{\varepsilon_r}}$$

$$\zeta = \sqrt{\mu_0 / (\varepsilon_0 \varepsilon_r)} = \zeta_0 / \sqrt{\varepsilon_r}$$

Plane waves in a conductor

Dielectric: σ=0 $\qquad \nabla \times \underline{H} = j\omega\varepsilon \underline{E}$

Conductor: σ≠0 $\qquad \nabla \times \underline{H} = j\omega\varepsilon \underline{E} + \sigma \underline{E} = j\omega\left(\varepsilon + \frac{\sigma}{j\omega}\right)\underline{E} = j\omega\varepsilon_{eff}\,\underline{E}$

$$\boxed{\varepsilon_{eff} = \varepsilon\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)}$$

$$\boxed{\begin{aligned} &E_x(z,\omega) = E_x^+ e^{-jkz} + E_x^- e^{jkz} \\ &E_x^+ = \zeta H_y^+ \quad E_x^- = -\zeta H_y^- \\ &H_y(z,\omega) = E_x^+/\zeta e^{-jkz} - E_x^-/\zeta e^{jkz} \end{aligned}}$$

$$\boxed{\begin{aligned} &E_y(z,\omega) = E_y^+ e^{-jkz} + E_y^- e^{jkz} \\ &E_y^+ = -\zeta H_x^+ \quad E_y^- = \zeta H_x^- \\ &H_x(z,\omega) = -E_y^+/\zeta e^{-jkz} + E_y^-/\zeta e^{jkz} \end{aligned}}$$

$$E_z = 0 \quad H_z = 0$$

$$\zeta = \sqrt{\mu/\varepsilon_{eff}} = \sqrt{\mu \Big/ \left[\varepsilon\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)\right]} = R + jX$$

$$k = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu\left[\varepsilon\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)\right]} = \beta - j\alpha$$

$\begin{cases} \beta\text{- } \textit{Phase constant} \\ \alpha\text{- } \textit{Attenuation constant} \end{cases}$

Conductor propagation constant

$$\begin{cases} \bullet \text{ characteristic impedance } \zeta = \sqrt{\dfrac{\mu}{\varepsilon_{eff}}} = \sqrt{\dfrac{\mu_0}{\varepsilon_0\varepsilon_r\left(1 - j\dfrac{\sigma_2}{\omega\varepsilon_0\varepsilon_r}\right)}} \\ \bullet \text{ propagation constant } k = \omega\sqrt{\mu\varepsilon_{eff}} = \omega\sqrt{\varepsilon_0\varepsilon_r\mu_0\left(1 - j\dfrac{\sigma}{\omega\varepsilon_0\varepsilon_r}\right)} = \beta - j\alpha \end{cases}$$

$$\left(\sqrt{z} = \sqrt{\frac{|z| + \mathrm{Re}(z)}{2}} - j\sqrt{\frac{|z| - \mathrm{Re}(z)}{2}}\right)$$

$$\beta = \omega\sqrt{\frac{\mu_0\varepsilon_0\varepsilon_r}{2}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon_0\varepsilon_r}\right)^2} + 1\right)} \quad \Rightarrow \quad \lambda = \frac{2\pi}{\beta} = \frac{c/f}{\sqrt{\dfrac{\varepsilon_r}{2}\sqrt{1 + \left(\dfrac{\sigma}{\omega\varepsilon_0\varepsilon_r}\right)^2} + 1}}$$

$$\alpha = \omega\sqrt{\frac{\mu_0\varepsilon_0\varepsilon_r}{2}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon_0\varepsilon_r}\right)^2} - 1\right)} \quad \Rightarrow \quad \boxed{\delta = 1/\alpha = \frac{c}{\omega\sqrt{\dfrac{\varepsilon_r}{2}\left(\sqrt{1 + \left(\dfrac{\sigma}{\omega\varepsilon_0\varepsilon_r}\right)^2} - 1\right)}}}$$

$$e^{-\alpha z}\left(z = \delta\right) = e^{-1} = 1/e$$ (about 0.37 or -8.69dB)

Good conductor

$$\boxed{\left|\frac{\sigma}{\omega\varepsilon_0\varepsilon_r}\right| >> 1}$$

propagation constant:

$$k = \omega\sqrt{\mu_0\varepsilon_0\varepsilon_r\left(1-\frac{j\sigma}{\omega\varepsilon_0\varepsilon_r}\right)} \cong \omega\sqrt{\frac{-j\sigma\mu_0}{\omega}} = \sqrt{\frac{\sigma\mu_0\omega}{2}}\sqrt{-2j}$$

characteristic impedance:

$$\zeta = \sqrt{\frac{\mu_0}{\varepsilon_0\varepsilon_r\left(1-j\frac{\sigma}{\omega\varepsilon_0\varepsilon_r}\right)}} \cong \sqrt{\frac{\mu_0}{\left(-\frac{j\sigma}{\omega}\right)}} = \sqrt{\frac{\omega\mu_0}{2\sigma}}\sqrt{2j}$$

$$(1-j)^2 = -2j \;\&\; (1+j)^2 = 2j \Longrightarrow \begin{cases} k \cong \sqrt{\dfrac{\sigma\mu_0\omega}{2}}(1-j) = \beta - j\alpha \qquad \beta = \alpha = \sqrt{\dfrac{\sigma\mu_0\omega}{2}} \\ \zeta \cong \sqrt{\dfrac{\mu_0\omega}{2\sigma}}(1+j) = \dfrac{1}{\sigma\delta}(1+j) = R_s(1+j) \end{cases}$$

$$\boxed{\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}}}$$

Good-conductor penetration depth

$$\boxed{R_s = \frac{1}{\sigma\delta}}$$

Good-conductor surface resistivity

Low losses material

$$\boxed{\left|\frac{\sigma}{\omega\varepsilon_0\varepsilon_r}\right| << 1}$$

$$\left\{\begin{array}{l} k = \omega\sqrt{\mu_0\varepsilon_0\varepsilon_r}\left(1 - \dfrac{j\sigma}{2\omega\varepsilon_0\varepsilon_r}\right) \cong \dfrac{\omega}{c}\sqrt{\varepsilon_r} - j\dfrac{\sigma}{2}\sqrt{\dfrac{\mu_0}{\varepsilon_0\varepsilon_r}} \\ \\ \zeta = \sqrt{\dfrac{\mu_0}{\varepsilon_0\varepsilon_r\left(1 - j\dfrac{\sigma}{\omega\varepsilon_0\varepsilon_r}\right)}} \cong \sqrt{\dfrac{\mu_0}{\varepsilon_0\varepsilon_r}}\left(1 + j\dfrac{\sigma}{2\omega\varepsilon_0\varepsilon_r}\right) \cong \zeta_0 / \sqrt{\varepsilon_r} \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{l} \lambda = \dfrac{2\pi}{\beta} \cong \dfrac{c/f}{\sqrt{\varepsilon_r}} \\ \\ \delta = \dfrac{1}{\alpha} \cong \dfrac{2}{\sigma}\sqrt{\dfrac{\varepsilon_0\varepsilon_r}{\mu_0}} \end{array}\right.$$

Penetration Depth: examples

Material	Frequency	Conductivity [S/m]	Depth penetration [mm]
Aluminum	100Hz	3.54*10^7	8.5
	10GHz	3.54*10^7	0.85*10^-3
Blood	900MHz	1.5379	27.8
	2.4GHz	2.5024	0.0164
Fat	900MHz	0.0510	244.12
	2.4GHz	0.10235	119.56
Sea water	300Hz	5	1300

*Penetration depth explains the **skin effect**: while the frequency increases, the penetration depth decreases, and the currents only flow on the conductor surface.*