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Electromagnetic Radiations and Biological Interactions

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Poynting's Theorem

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Lecture Content

- Poynting vector
- Poynting's theorem

- Poynting vector – definition in time domain
- Poynting's theorem - Time domain (instantaneous-power balance equation)

- Poynting vector – definition in frequency domain
- Poynting's theorem- Frequency domain (average-power balance equation)

- Poynting vector associated to an electromagnetic plane wave
- Examples and discussion

Poynting's Theorem – Time domain

The Poynting's theorem: It is a statement about the conservation of energy for the electromagnetic field. **It comes from Maxwell's equations.**

$$\underline{s}(\underline{r},t)=\underline{e}(\underline{r},t)\times\underline{h}(\underline{r},t)\quad [\mathrm{W/m^2}]$$

Poynting vector

$$\left(\nabla\cdot(\underline{a}\times\underline{b})=\underline{b}\cdot(\nabla\times\underline{a})-\underline{a}\cdot(\nabla\times\underline{b})\right)\qquad \nabla\cdot\underline{s}=\nabla\cdot(\underline{e}\times\underline{h})=\underline{h}\cdot\nabla\times\underline{e}-\underline{e}\cdot\nabla\times\underline{h}$$

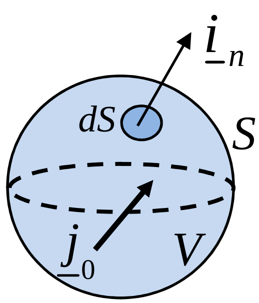
$$\nabla\times\underline{e}(\underline{r},t)=-\frac{\partial}{\partial t}\underline{b}(\underline{r},t)$$

$$\nabla\times\underline{h}(\underline{r},t)=\frac{\partial}{\partial t}\underline{d}(\underline{r},t)+\underline{j}(\underline{r},t)$$

$$\underline{j}=\sigma\underline{e}+\underline{j}_0$$

$\underline{j}_0$ → current density due to the sources

$$\Rightarrow \nabla\cdot\underline{s}=-\underline{h}\cdot\frac{\partial\underline{b}}{\partial t}-\underline{e}\cdot\frac{\partial\underline{d}}{\partial t}-\underline{e}\cdot\underline{j}\ \Rightarrow$$



$$\iiint_V \Rightarrow \quad \iiint_V\nabla\cdot\underline{s}\,dV+\iiint_V\left(\underline{h}\cdot\frac{\partial\underline{b}}{\partial t}+\underline{e}\cdot\frac{\partial\underline{d}}{\partial t}\right)dV+\iiint_V\sigma\left|\underline{e}\right|^2dV=-\iiint_v\underline{e}\cdot\underline{j}_0\,dV$$

$$\oiint_S\underline{A}\cdot\underline{i}_n\,dS=\iiint_V(\nabla\cdot\underline{A})\,dV\qquad \text{(Divergence theorem)}$$

$\Rightarrow$

$$\oiint_S\underline{s}\cdot\underline{i}_n\,dS+\iiint_V\left(\underline{h}\cdot\frac{\partial\underline{b}}{\partial t}+\underline{e}\cdot\frac{\partial\underline{d}}{\partial t}\right)dV+\iiint_V\sigma e^2dV=-\iiint_v\underline{e}\cdot\underline{j}_0\,dV\quad \text{[W (t), instantaneous power]}$$

Poynting's theorem

Poynting's Theorem – Physical meaning

$$\oiint_S \underline{s}\cdot\underline{i}_n dS + \iiint_V \left(\underline{h}\cdot\frac{\partial \underline{b}}{\partial t} + \underline{e}\cdot\frac{\partial \underline{d}}{\partial t}\right) dV + \iiint_V \sigma e^2 dV = -\iiint_V \underline{e}\cdot\underline{j}_0 dV$$ ***Poynting's theorem***

Interpretation of: $-\iiint_V \underline{e}\cdot\underline{j}_0 dV$

Lorentz force equation

- Consider the current density $\underline{j}_0$ created by the charge density ρ_0 that moves with a velocity $\underline{v}$: $\underline{j}_0 = \rho_0 \underline{v}$
- $\underline{j}_0$ generates an electromagnetic field and a force density acts on the source field: $\underline{f} = \rho_0 \underline{e} + \rho_0 \underline{v}\times\underline{b}$ [N/m³]
- In order to keep the velocity $\underline{v}$ of charge density ρ_0, a force density $-\underline{f} = -\rho_0 \underline{e} - \rho_0 \underline{v}\times\underline{b}$ must be applied.
- For an elementary displacement $d\underline{r}$: $-\underline{f}\cdot d\underline{r} = -\rho_0 \underline{e}\cdot d\underline{r} - \rho_0 \underline{v}\times\underline{b}\cdot d\underline{r}$
- in a time interval dt: $\frac{d\underline{r}}{dt} = \underline{v}$

$\Rightarrow$

$$-\underline{f}\cdot\underline{v} = -\underline{e}\cdot\underline{j}_0 - \rho_0 \underline{v}\times\underline{b}\cdot\underline{v} = -\underline{e}\cdot\underline{j}_0 \xrightarrow{\iiint_V}$$

$-\iiint_V \underline{e}\cdot\underline{j}_0 dV$ → Represents the instantaneous power delivered by the sources $\underline{j}_0$

Poynting's Theorem – Physical meaning

Interpretation of: $\iiint_V \sigma e^2 dV$ → Represents the instantaneous power dissipated by Joule effect in a volume V

Interpretation of: $\iiint_V \left(\underline{h} \cdot \frac{\partial \underline{b}}{\partial t} + \underline{e} \cdot \frac{\partial \underline{d}}{\partial t} \right) dV$

Medium: linear, homogeneous, isotropic and non-dispersive in time: $\underline{d} = \varepsilon \underline{e}$ & $\underline{b} = \mu \underline{h}$

$$\Rightarrow \quad \underline{e} \cdot \frac{\partial \underline{d}}{\partial t} = \underline{e} \cdot \varepsilon \frac{\partial \underline{e}}{\partial t} = \varepsilon \frac{\partial}{\partial t}\left(\frac{1}{2} \underline{e} \cdot \underline{e} \right) = \frac{\partial}{\partial t}\left(\frac{1}{2} \varepsilon \left| \underline{e} \right|^2 \right) \quad \& \quad \underline{h} \cdot \frac{\partial \underline{b}}{\partial t} = \underline{h} \cdot \mu \frac{\partial \underline{h}}{\partial t} = \mu \frac{\partial}{\partial t}\left(\frac{1}{2} \underline{h} \cdot \underline{h} \right) = \frac{\partial}{\partial t}\left(\frac{1}{2} \mu \left| \underline{h} \right|^2 \right)$$

For static fields: $\frac{1}{2} \varepsilon \left| \underline{e} \right|^2$ → electrostatic field energy & $\frac{1}{2} \mu \left| \underline{h} \right|^2$ → magnetostatic field energy

$\underline{e} \cdot \frac{\partial \underline{d}}{\partial t} = \frac{\partial w_e}{\partial t}$ & $\underline{h} \cdot \frac{\partial \underline{b}}{\partial t} = \frac{\partial w_m}{\partial t}$ → Represent the time variation of the magnetic and electric energies stored in V

$$w = w_e + w_m = \frac{1}{2} \varepsilon \left| \underline{e} \right|^2 + \frac{1}{2} \mu \left| \underline{h} \right|^2 \quad \text{[Joule/m}^3\text{]} \quad \overset{\iiint_V}{\Longrightarrow}$$

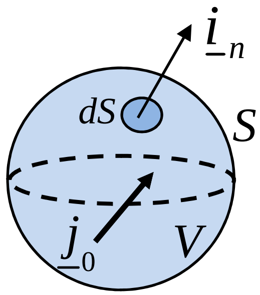
$$\iiint_V \left(\underline{h} \cdot \frac{\partial \underline{b}}{\partial t} + \underline{e} \cdot \frac{\partial \underline{d}}{\partial t} \right) dV = \frac{\partial}{\partial t} \iiint_V (w_e + w_m) dV \quad \text{[W (t), instantaneous power]}$$

→ Represents time variation of the electric and magnetic field energies stored in a volume V.

Poynting's Theorem – Physical meaning

$$\oiint_S \underline{s}\cdot\underline{i}_n dS + \iiint_V \left(\underline{h}\cdot\frac{\partial \underline{b}}{\partial t} + \underline{e}\cdot\frac{\partial \underline{d}}{\partial t}\right) dV + \iiint_V \sigma e^2 dV = -\iiint_V \underline{e}\cdot\underline{j}_0 dV$$ ***Poynting's theorem***

Interpretation of: $\oiint_S \underline{s}\cdot\underline{i}_n dS$ → Represents the instantaneous power flux through the closed surface S.



➡ The power generated by the sources in a volume V is equal to the sum of the power flowing through the surface *S*, the time-variation of the electromagnetic energy stored in *V* and the power dissipated into the volume *V* (instantaneous powers).

Notes

When considering an arbitrary periodic excitation:

$$p(t) = v(t)i(t)$$ instantaneous power

$$< p(t) >= \frac{1}{T}\int_0^T v(t)i(t)dt$$ average (or active) power

$$w = \int_0^{t_0} p(t)dt, \quad p(t) = \frac{dw(t)}{dt}$$

$$\begin{cases} w_c(t) = \frac{1}{2}Cv^2(t) & \text{If C is the capacitance of a capacitor} \\ < \frac{1}{2}Cv^2(t) >= \frac{1}{T}\int_0^T \frac{1}{2}Cv^2(t)dt \end{cases}$$

- For a sinusoidal excitation: $T = 2\pi / \omega$

$$< p(t) >= \frac{1}{T}\int_0^T v(t)i(t)dt = \mathrm{Re}\left\{\frac{1}{2}VI^*\right\}$$

$$< w_c(t) >= \frac{1}{4}C|V|^2$$

Phasor Arithmetic

Consider the following phasors: $\underline{f} = \mathrm{Re}\left\{\underline{F}e^{j\omega t}\right\} = \dfrac{\underline{F}e^{j\omega t} + \underline{F}^* e^{-j\omega t}}{2}$

$$\underline{g} = \mathrm{Re}\left\{\underline{G}e^{j\omega t}\right\} = \frac{\underline{G}e^{j\omega t} + \underline{G}^* e^{-j\omega t}}{2}$$

Scalar product: $\underline{f}\cdot\underline{g} = \dfrac{\underline{F}e^{j\omega t} + \underline{F}^* e^{-j\omega t}}{2}\cdot\dfrac{\underline{G}e^{j\omega t} + \underline{G}^* e^{-j\omega t}}{2} = \dfrac{\underline{F}\cdot\underline{G}^* + \underline{F}^*\cdot\underline{G}}{4} + \dfrac{\underline{F}\cdot\underline{G}e^{2j\omega t} + \underline{F}^*\cdot\underline{G}e^{2j\omega t}}{4} = \mathrm{Re}\left\{\dfrac{\underline{F}\cdot\underline{G}^*}{2}\right\} + \mathrm{Re}\left\{\dfrac{\underline{F}\cdot\underline{G}}{2}e^{2j\omega t}\right\}$

Average scalar product: $\left\langle \underline{f}\cdot\underline{g}\right\rangle = \dfrac{1}{T}\int_0^T \underline{f}\cdot\underline{g}\,dt = \mathrm{Re}\left\{\dfrac{\underline{F}\cdot\underline{G}^*}{2}\right\}$

If: $\underline{g}_1 = \dfrac{\partial \underline{g}}{\partial t} \Rightarrow \left\langle \underline{f}\cdot\underline{g}_1\right\rangle = \mathrm{Re}\left\{-j\omega\dfrac{1}{2}\underline{F}\cdot\underline{G}^*\right\}$

Average vector product: $\left\langle \underline{f}\times\underline{g}\right\rangle = \mathrm{Re}\left\{\dfrac{1}{2}\underline{F}\times\underline{G}^*\right\}$

Poynting's Theorem – Frequency domain

$$\underline{e}(\underline{r},t) \xrightarrow{phasor} \underline{E}(\underline{r})$$
$$\underline{h}(\underline{r},t) \xrightarrow{phasor} \underline{H}(\underline{r})$$

$$\underline{S} = \frac{1}{2}\underline{E}\times\underline{H}^* = \underline{S}_r + j\underline{S}_i$$ **Poynting vector** $\Rightarrow$

$$\left(\nabla\cdot(\underline{a}\times\underline{b}) = \underline{b}\cdot(\nabla\times\underline{a}) - \underline{a}\cdot(\nabla\times\underline{b})\right)$$

$$\nabla\cdot\underline{S} = \frac{1}{2}\left[\underline{H}^*\cdot\nabla\times\underline{E} - \underline{E}\cdot\nabla\times\underline{H}^*\right]$$

$$\underline{J} = \sigma\underline{E} + \underline{J}_0$$

From the first two Maxwell's equations:

$$\begin{cases}\nabla\times\underline{E}(\underline{r},\omega) = -j\omega\underline{B}(\underline{r},\omega)\\ \nabla\times\underline{H}(\underline{r},\omega) = j\omega\underline{D}(\underline{r},\omega) + \underline{J}(\underline{r},\omega)\end{cases}$$

In a linear, isotropic, lossy and dispersive medium:

$$\underline{D}(\underline{r},\omega) = \varepsilon(\omega)\underline{E}(\underline{r},\omega)$$
$$\underline{B}(\underline{r},\omega) = \mu(\omega)\underline{H}(\underline{r},\omega)$$
$$\varepsilon(\omega) = \varepsilon'(\omega) - j\varepsilon''(\omega)$$
$$\mu(\omega) = \mu'(\omega) - j\mu''(\omega)$$

$\Rightarrow$

$$\nabla\times\underline{E} = -j\omega\left(\mu' - j\mu''\right)\underline{H}$$
$$\boxed{\nabla\times\underline{H} = j\omega(\varepsilon' - j\varepsilon'')\underline{E} + \sigma\underline{E} + \underline{J}_0}$$

$\downarrow *$

$$\nabla\times\underline{H}^* = -j\omega(\varepsilon' + j\varepsilon'')\underline{E}^* + \sigma\underline{E}^* + \underline{J}_0^*$$

$$\Rightarrow \boxed{\nabla\cdot\underline{S} + j\omega\left(\frac{\mu'|\underline{H}|^2}{2} - \frac{\varepsilon'|\underline{E}|^2}{2}\right) + \frac{\omega\mu''|\underline{H}|^2}{2} + \frac{\omega\varepsilon''|\underline{E}|^2}{2} + \frac{\sigma|\underline{E}|^2}{2} = -\frac{1}{2}\underline{E}\cdot\underline{J}_0^*}$$

Poynting's theorem (differential form)

Poynting's Theorem – Frequency domain

$$\nabla \cdot \underline{S} + j\omega\left(\frac{\mu'\left|\underline{H}\right|^2}{2} - \frac{\varepsilon'\left|\underline{E}\right|^2}{2}\right) + \frac{\omega\mu''\left|\underline{H}\right|^2}{2} + \frac{\omega\varepsilon''\left|\underline{E}\right|^2}{2} + \frac{\sigma\left|\underline{E}\right|^2}{2} = -\frac{1}{2}\underline{E}\cdot\underline{J}_0^* \qquad \iiint_V \Longrightarrow$$

Poynting's theorem (integral form)

$$\oiint_S \underline{S}\cdot\underline{i}_n dS + \iiint_V\left(\frac{\omega\mu''\left|\underline{H}\right|^2}{2} + \frac{\omega\varepsilon''\left|\underline{E}\right|^2}{2}\right)dV + j2\omega\iiint_V\left(\frac{\mu'\left|\underline{H}\right|^2}{4} - \frac{\varepsilon'\left|\underline{E}\right|^2}{4}\right)dV + \iiint_V \frac{\sigma\left|\underline{E}\right|^2}{2}dV = -\iiint_V \frac{1}{2}\underline{E}\cdot\underline{J}_0^* dV$$

$$\underline{S} = \frac{1}{2}\underline{E}\times\underline{H}^* = \underline{S}_r + j\underline{S}_i$$

Real part:

$$\nabla\cdot\underline{S}_r + \frac{\omega\mu''\left|\underline{H}\right|^2}{2} + \frac{\omega\varepsilon''\left|\underline{E}\right|^2}{2} + \frac{\sigma\left|\underline{E}\right|^2}{2} = -\mathrm{Re}\left\{\frac{1}{2}\underline{E}\cdot\underline{J}_0^*\right\}$$

Imaginary part:

$$\nabla\cdot\underline{S}_i + 2\omega\left(\frac{\mu'\left|\underline{H}\right|^2}{4} - \frac{\varepsilon'\left|\underline{E}\right|^2}{4}\right) = -\mathrm{Im}\left\{\frac{1}{2}\underline{E}\cdot\underline{J}_0^*\right\}$$

Poynting's Theorem – Average power balance equation

$$\nabla \cdot \underline{S}_r + \frac{\omega\mu''|\underline{H}|^2}{2} + \frac{\omega\varepsilon''|\underline{E}|^2}{2} + \frac{\sigma|\underline{E}|^2}{2} = -\mathrm{Re}\left\{\frac{1}{2}\underline{E}\cdot\underline{J}_0^*\right\}$$

Considering the phasor arithmetic: If: $g_1 = \frac{\partial g}{\partial t} \Rightarrow \langle \underline{f} \cdot \underline{g}_1 \rangle = \mathrm{Re}\left\{-j\omega\frac{1}{2}\underline{F}\cdot\underline{G}^*\right\}$

$$\frac{\sigma|\underline{E}|^2}{2} = \mathrm{Re}\left\{\frac{\sigma\underline{E}\cdot\overline{E}^*}{2}\right\} = \langle \sigma\underline{e}\cdot\underline{e}\rangle = \left\langle \sigma|\underline{e}|^2\right\rangle$$ → Average power dissipated by Joule effect.

$$\left\langle \underline{e}\cdot\frac{\partial \underline{d}}{\partial t}\right\rangle = \mathrm{Re}\left\{-\frac{j\omega}{2}\underline{E}\cdot\underline{D}^*\right\} = \mathrm{Re}\left\{-\frac{j\omega}{2}\underline{E}\cdot\left[(\varepsilon' - j\varepsilon'')\underline{E}\right]^*\right\} = \mathrm{Re}\left\{-\frac{j\omega}{2}\underline{E}\cdot\left[(\varepsilon' + j\varepsilon'')\underline{E}^*\right]\right\} = \frac{\omega\varepsilon''|\underline{E}|^2}{2}$$

$$\left\langle \underline{h}\cdot\frac{\partial \underline{b}}{\partial t}\right\rangle = \frac{\omega\mu''|\underline{H}|^2}{2}$$

They represent dissipated power (dielectric and magnetic polarization losses)

Poynting's Theorem – Reactive power balance equation

$$\nabla \cdot \underline{S}_i + 2\omega\left(\frac{\mu'\left|\underline{H}\right|^2}{4} - \frac{\varepsilon'\left|\underline{E}\right|^2}{4}\right) = -\operatorname{Im}\left\{\frac{1}{2}\underline{E}\cdot\underline{J}_0^*\right\}$$

$$\left\langle \frac{1}{2}\underline{e}\cdot\underline{d}\right\rangle = \frac{1}{2}\left\langle \underline{e}\cdot\underline{d}\right\rangle = \frac{1}{2}\operatorname{Re}\left\{\frac{1}{2}\underline{E}\cdot\underline{D}^*\right\} = \frac{1}{2}\operatorname{Re}\left\{\frac{1}{2}\underline{E}\cdot\left[\left(\varepsilon' + j\varepsilon''\right)\underline{E}^*\right]\right\} = \frac{\varepsilon'\left|\underline{E}\right|^2}{4}$$

$$\left\langle \frac{1}{2}\underline{e}\cdot\underline{d}\right\rangle = \frac{\varepsilon'\left|\underline{E}\right|^2}{4} = \left\langle \frac{1}{2}\varepsilon\left|\underline{e}\right|^2\right\rangle = \left\langle w_e\right\rangle$$

$$\left\langle \frac{1}{2}\underline{h}\cdot\underline{b}\right\rangle = \frac{\mu'\left|\underline{H}\right|^2}{4} = \left\langle \frac{1}{2}\mu\left|\underline{h}\right|^2\right\rangle = \left\langle w_m\right\rangle$$

They represent the average electric and magnetic energy stored per unit of volume Joule/m^3

Poynting's Theorem – Average power balance equation

$$\nabla \cdot \underline{S}_r + \frac{\omega\mu''|\underline{H}|^2}{2} + \frac{\omega\varepsilon''|\underline{E}|^2}{2} + \frac{\sigma|\underline{E}|^2}{2} = -\mathrm{Re}\left\{\frac{1}{2}\underline{E}\cdot\underline{J}_0^*\right\} \quad \iiint_V \Longrightarrow$$

$$\Longrightarrow \quad \oiint_S \underline{S}_r \cdot \underline{i}_n dS + \iiint_V \left(\frac{\omega\mu''|\underline{H}|^2}{2} + \frac{\omega\varepsilon''|\underline{E}|^2}{2}\right) dV + \iiint_V \frac{\sigma|\underline{E}|^2}{2} dV = P_r$$

- P_r: (average power delivered by the sources inside volume *V*)
- $\iiint_V \frac{\sigma|\underline{E}|^2}{2} dV$: average power dissipated by Joule effect inside V
- $\iiint_V \left(\frac{\omega\mu''|\underline{H}|^2}{2} + \frac{\omega\varepsilon''|\underline{E}|^2}{2}\right) dV$: average power associated to dielectric and magnetic polarization losses
- $\oiint_S \underline{S}_r \cdot \underline{i}_n dS$: the average power flux through the surface *S*

$\oiint_S \underline{A}\cdot\underline{i}_n dS = \iiint_V (\nabla\cdot\underline{A})\, dV$ (divergence theorem)

$$\frac{\sigma|\underline{E}|^2}{2} = \left\langle \sigma|\underline{e}|^2 \right\rangle, \quad \left\langle \underline{e}\cdot\frac{\partial \underline{d}}{\partial t}\right\rangle = \frac{\omega\varepsilon''|\underline{E}|^2}{2}, \quad \left\langle \underline{h}\cdot\frac{\partial \underline{b}}{\partial t}\right\rangle = \frac{\omega\mu''|\underline{H}|^2}{2}$$

$-\iiint_V \frac{1}{2}\underline{E}\cdot\underline{J}_0^* dV = P_r + jP_i$ (average power and reactive power delivered by the sources inside volume *V*)

Poynting's Theorem – Reactive power balance equation

$$\nabla \cdot \underline{S}_i + 2\omega\left(\frac{\mu'|\underline{H}|^2}{4} - \frac{\varepsilon'|\underline{E}|^2}{4}\right) = -\operatorname{Im}\left\{\frac{1}{2}\underline{E}\cdot\underline{J}_0^*\right\}$$

(reactive power exchanged by the sources inside V)

$$\Rightarrow \oiint_S \underline{S}_i \cdot \underline{i}_n dS + 2\omega\iiint_V \left(\frac{\mu'|\underline{H}|^2}{4} - \frac{\varepsilon'|\underline{E}|^2}{4}\right) dV = P_i$$

$$\frac{1}{2\omega}\oiint_S \underline{S}_i \cdot \underline{i}_n dS + \iiint_V \left(\frac{\mu'|\underline{H}|^2}{4} - \frac{\varepsilon'|\underline{E}|^2}{4}\right) dV = \frac{P_i}{2\omega}$$

Represents an energy exchange between sources and external environment .

Average electric and magnetic energies stored in a volume V

the average energy exchanged through the surface *S*

$$\langle w_e \rangle = \frac{1}{2}\langle \underline{e}\cdot\underline{d}\rangle = \frac{\varepsilon'|\underline{E}|^2}{4}, \quad \langle w_m \rangle = \frac{1}{2}\langle \underline{h}\cdot\underline{b}\rangle = \frac{\mu'|\underline{H}|^2}{4}$$

At a resonance frequency: $\iiint_V \frac{\mu'|\underline{H}|^2}{4} dV = \iiint_V \frac{\varepsilon'|\underline{E}|^2}{4} dV \qquad \frac{P_i}{2\omega} = \frac{1}{2\omega}\oiint_S \underline{S}_i \cdot \underline{i}_n dS$

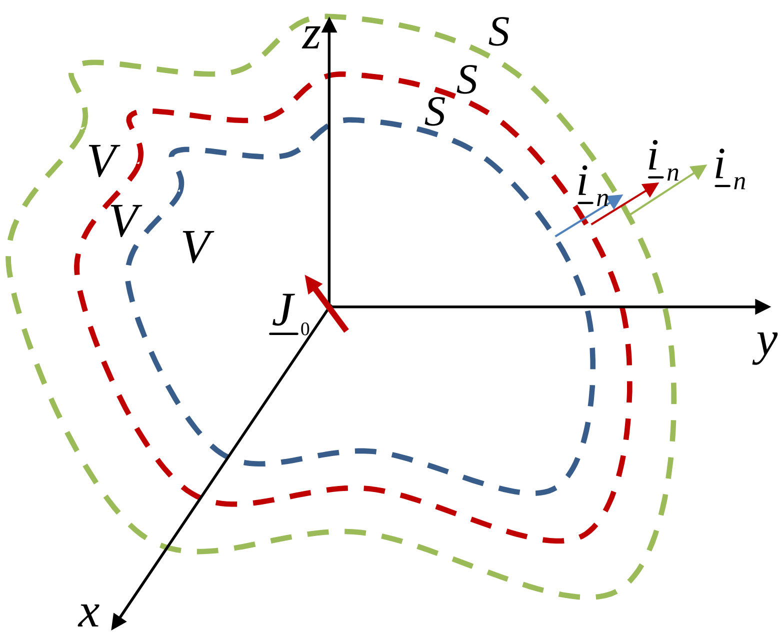
Poynting's Theorem – Average power balance equation

Consider a case with no losses: $\iiint_V \left(\frac{\omega\mu''|\underline{H}|^2}{2} + \frac{\omega\varepsilon''|\underline{E}|^2}{2} \right) dV = 0$

$\iiint_V \frac{\sigma|\underline{E}|^2}{2} dV = 0$

$\Rightarrow \oiint_S \underline{S}_r \cdot \underline{i}_n dS = P_r$ *Active power*

The average power generated by the field sources can be interpreted as the flux of the real part of the Poynting's vector through any surface S surrounding the source.



Increasing the surface S, P_r *doesn't change* $\Rightarrow$ *the flux of* $\underline{S}_r$ *is independent of the surface* S.

$\Downarrow$

Transmitting antenna: there is a power available from the sources with a lower density at larger distances (E~1/r):

$$P(r) = \oiint_{S(r)} \underline{S}_r \cdot \underline{i}_n dS = \int_0^{\pi}\int_0^{2\pi} \frac{1}{2\zeta_0}|\underline{E}(r)|^2 dS =$$

$$\int_0^{\pi}\int_0^{2\pi} \frac{1}{2\zeta_0}|\underline{E}_0|^2 \left(\frac{1}{r}\right)^2 r^2 \sin\theta d\theta d\phi = const. = P_{radiated}$$

Poynting's Vector of a Plane Wave

$$\underline{E} = \underline{E}_0 e^{-jkz}$$
$$\underline{H} = \underline{H}_0 e^{-jkz} = \frac{1}{\zeta}\underline{i}_z \times \underline{E}_0 e^{-jkz}$$
$$k = \beta - j\alpha \;\&\; \zeta = \sqrt{\frac{\mu}{\varepsilon}}$$

$$\Rightarrow \underline{S} = \frac{1}{2}\underline{E}\times\underline{H}^* = \frac{1}{2\zeta^*}\underline{E}_0 \times \left(\underline{i}_z \times \underline{E}_0^*\right) e^{-jkz} e^{jk^* z}$$
$$\underline{A}\times\left(\underline{B}\times\underline{C}\right) = \underline{B}\cdot\left(\underline{A}\cdot\underline{C}\right) - \underline{C}\cdot\left(\underline{A}\cdot\underline{B}\right)$$
$$\underline{i}_z \cdot \underline{E}_0 = 0$$
$$\underline{E}_0 \cdot \underline{E}_0^* = \left|\underline{E}_0\right|^2$$

$$\Rightarrow \underline{S} = \frac{1}{2\zeta^*}\left|\underline{E}_0\right|^2 e^{-2\alpha z}\underline{i}_z$$

If the medium is lossless: $\underline{S} = \frac{1}{2\zeta}\left|\underline{E}_0\right|^2 \underline{i}_z = S\underline{i}_z, \quad S = \frac{1}{2\zeta}\left|\underline{E}_0\right|^2 = \frac{1}{2}\zeta\left|\underline{H}_0\right|^2$

In a lossless medium (non dissipative):
- $\underline{S}$ has the same direction as the wave propagation
- $\underline{S} = S\underline{i}_z$ with S real and constant with respect to z

In a medium with no losses and an arbitrary propagation direction: $\underline{S} = \frac{1}{2\zeta}\left|\underline{E}_0\right|^2 \underline{i} = \frac{1}{2}\zeta\left|\underline{H}_0\right|^2 \underline{i}$

In a lossy medium:
- $\underline{S}$ has the same direction as the wave propagation
- $\underline{S} = S\underline{i}_z$ with S having both a real and an imaginary part
- S decreases as $e^{-2\alpha z}$

$$\mathrm{Re}\{\underline{S}\} = \frac{1}{2}\mathrm{Re}\left\{\frac{1}{\zeta^*}\right\}\left|\underline{E}_0\right|^2 e^{-2\alpha z}\underline{i}_z = \frac{1}{2}\mathrm{Re}\{\zeta^*\}\left|\underline{H}_0\right|^2 e^{-2\alpha z}\underline{i}_z$$

Two waves propagating in opposite directions

$$\begin{cases} \underline{E} = \left[E_x^+ e^{-j\beta z} + E_x^- e^{+j\beta z}\right]\underline{i}_x \\ \underline{H} = \left[\dfrac{E_x^+}{\zeta_1} e^{-j\beta z} - \dfrac{E_x^-}{\zeta_1} e^{+j\beta z}\right]\underline{i}_y \end{cases}$$

$$\underline{S} = \frac{1}{2}\underline{E}\times\underline{H}^* = \frac{1}{2}\left[\frac{\left|E_x^+\right|^2}{\zeta} - \frac{\left|E_x^-\right|^2}{\zeta} - \frac{\left(E_x^+\right)\left(E_x^-\right)^*}{\zeta} e^{-j2\beta z} + \frac{\left(E_x^-\right)\left(E_x^+\right)^*}{\zeta} e^{j2\beta z}\right]\underline{i}_z$$

$$\left(E_x^+\right)\left(E_x^-\right)^* e^{-j2\beta z} = \left[\left(E_x^+\right)^*\left(E_x^-\right)e^{j2\beta z}\right]^*$$

$$\left(a+jb\right)-\left(a+jb\right)^* = 2jb$$

$\Rightarrow$

$$\mathrm{Re}\{\underline{S}\} = \frac{1}{2\zeta}\left[\left(E_x^+\right)^2 - \left(E_x^-\right)^2\right]\underline{i}_z = \left(\frac{1}{2\zeta}\left|E_x^+\right|^2 - \frac{1}{2\zeta}\left|E_x^-\right|^2\right)\underline{i}_z = \frac{1}{2\zeta}\left|E_x^+\right|^2\left(1-\left|\Gamma\right|^2\right)\underline{i}_z$$

$$\Gamma = \frac{E_x^-}{E_x^+}$$